Curves and surfaces & modeling case studies

Karan Singh

Dynamic Graphics Project
University of Toronto
www.dgp.toronto.edu

Curves and Surfaces

- Polynomial curves from constraints: Hermites.
- Basis functions: Beziers, BSplines.
- Coons Interpolation: Coons patches.
- Desirable curve properties.
- Other curve formulations: clothoids.

Parametric Polynomial Curves

Recall a linear curve (line) is: $p(t) = a_1 t + a_0$

A cubic curve is similarly:

$$
\begin{aligned}
x(t) &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\
y(t) &= b_3 t^3 + b_2 t^2 + b_1 t + b_0 \\
z(t) &= c_3 t^3 + c_2 t^2 + c_1 t + c_0
\end{aligned}
$$

…or $p(t) = d_3 t^3 + d_2 t^2 + d_1 t + d_0$, where $d_i = [a_i, b_i, c_i,]^T$

Cubics are commonly used in graphics because:

- curves of lower order have too little flexibility (only planar, no curvature control).
- curves of higher order are unnecessarily complex and easily wiggle.

Polynomial curves from constraints

$p(t) = TA$, where T is powers of t. for a cubic $T=[t^3\ t^2\ t^1\ 1]$.

Written with geometric constraints $p(t) = TMG$,
where M is the **Basis matrix** of the curve, G the design constraints.

An example of constraints for a cubic Hermite for eg. are
end points and end tangents. i.e. P_1,R_1 at $t=0$ and P_4,R_4 at $t=1$.
Plugging these constraints into $p(t) = TA$ we get.

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad H \\
p(0) = P_1 = [\ 0\ 0\ 0\ 1\ ]\ A \\
p(1) = P_4 = [\ 1\ 1\ 1\ 1\ ]\ A \qquad\qquad TA=TMG\ \& \\
p'(0)= R_1 = [\ 0\ 0\ 1\ 0\ ]\ A \quad => \quad G=HA \qquad => H=M^{-1} \\
p'(1)= R_4 = [\ 3\ 2\ 1\ 0\ ]\ A
\end{array}
$$

Bezier Basis Matrix

A cubic Bezier can be defined with four points where:
P_1,R_1 at t=0 and P_4,R_4 at t=1 for a Hermite.
$R_1 = 3(P_2-P_1)$ and $R_4 = 3(P_4-P_3)$.

We can thus compute the Bezier Basis Matrix by finding the matrix that transforms $[P_1\ P_2\ P_3\ P_4\ ]^T$ into $[P_1\ P_4\ R_1\ R_4\ ]^T$ i.e.

B_H = [1 0 0 0]
[0 0 0 1]
[-3 3 0 0]
[0 0 -3 3]

$M_{bezier} = M_{hermite} * B\_H$

Bezier Basis Functions

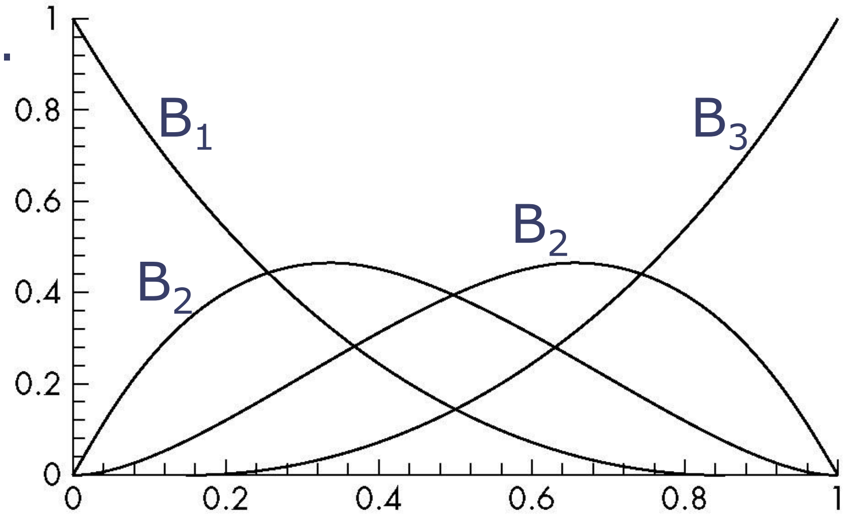
$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the Basis Matrix form Basis Functions such that:
$p(t)= B_1(t)P_1 + B_2(t)P_2 + B_3(t)P_3 + B_4(t)P_4.$

From the matrix:

$B_{i+1}(t) = \binom{n}{i} * (1-t)^{(n-i)} * t^i$

…also called Bernstein polynomials.



Basis Functions

Basis functions can be thought of as an influence weight that each constraint has as t varies.

Note: actual interpolation of any constraint only happens if its Basis function is 1 and all others are zero at some t.

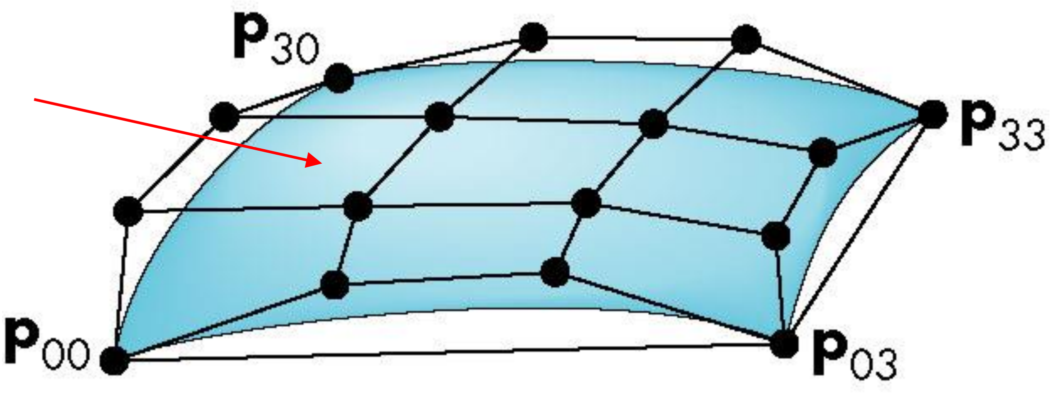
Often Basis functions for design curves sum to 1 for all t.
This gives the curve some nice properties like affine invariance and the convex hull property when the functions are additionally non-negative.

Bezier Patches

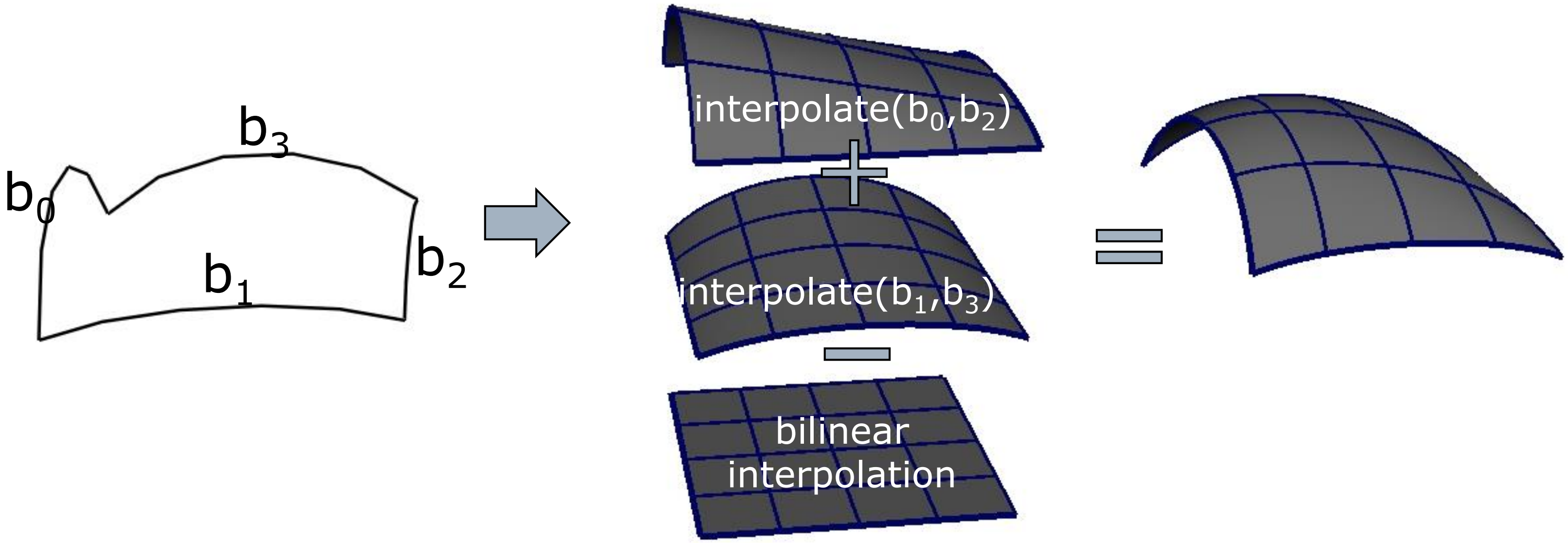
Using same data array $\mathbf{P}=[p_{ij}]$ as with interpolating form:

$$p(u,v) = \Sigma_i \Sigma_j B_i(u)\, B_j(v)\, p_{ij} = u^T\, \mathbf{B}\, \mathbf{P}\, \mathbf{B}^T\, v$$

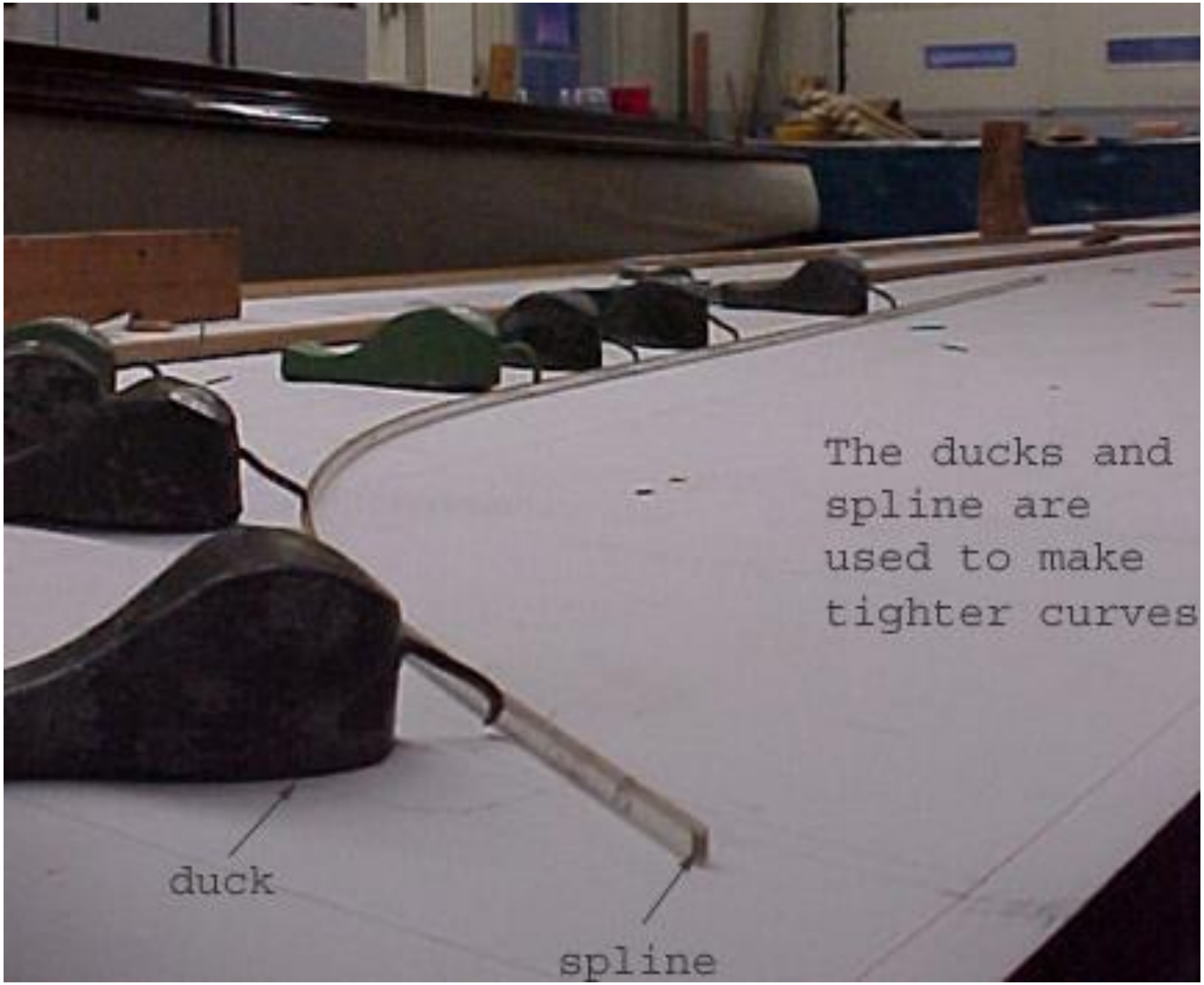
Patch lies in convex hull



Coons Patches: only boundary curves

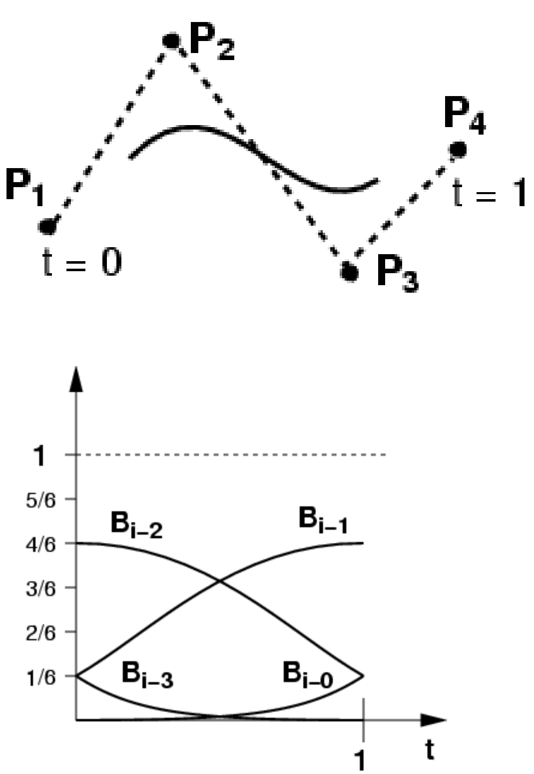


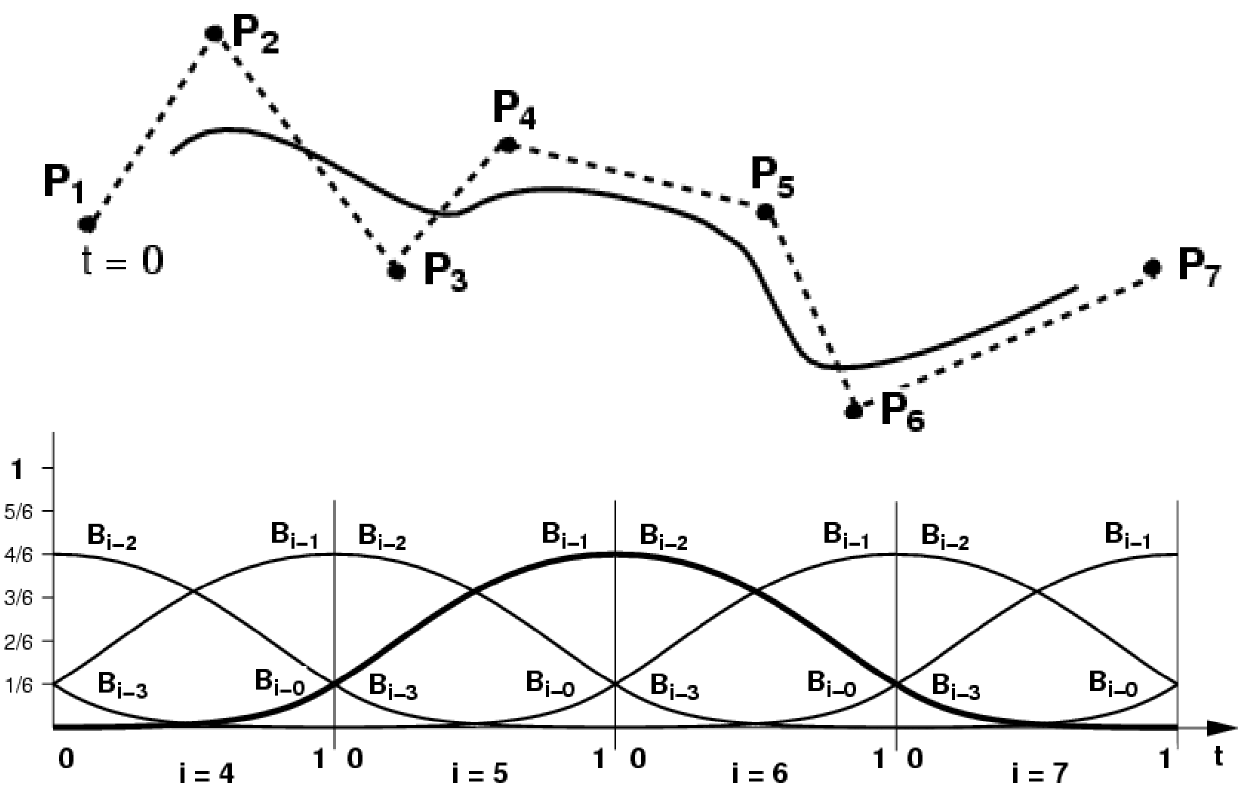
Traditional Splines



BSpline Basis Functions

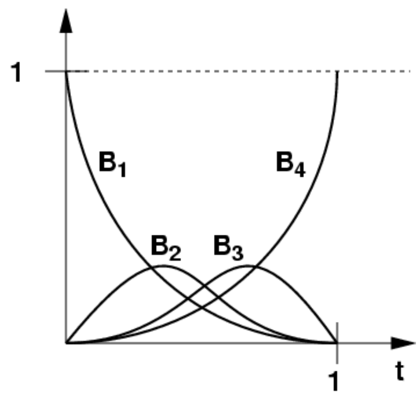
- Can be chained together.
- Local control (windowing).

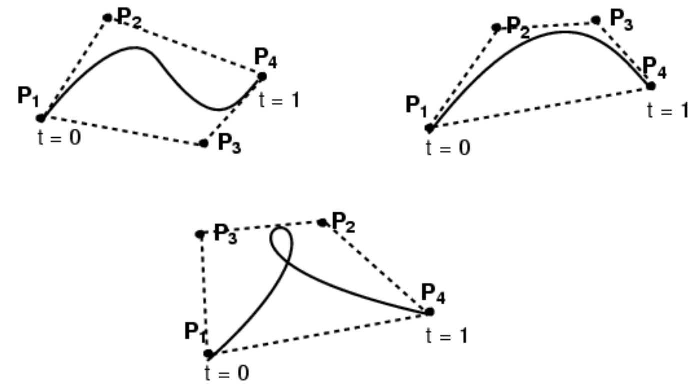




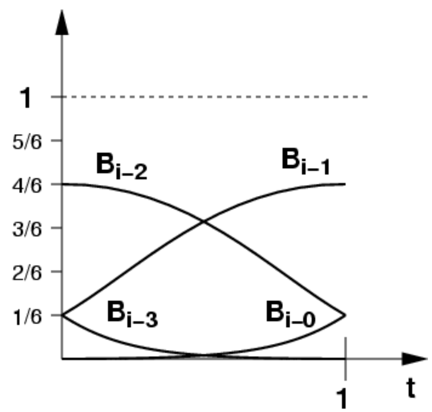
Bezier vs. BSpline

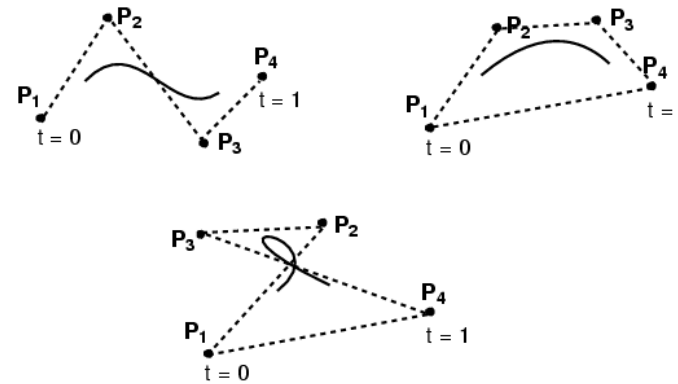
Bezier





BSpline





Representing a conic as a polynomial

- $\langle x(t),y(t)\rangle = \langle \cos(t), \sin(t) \rangle$
- Taylor series for $\sin(t)= t - t^3/3! + t^5/5! \ldots$
- $u=\sin(t/2)$
- $\langle x(u),y(u)\rangle = \langle (1-u^2)/(1+u^2)\ ,\ 2u/(1+u^2) \rangle$
- Rational Bspline’s are defined with homogeneous coordinates using w(t).
- NURBS additionally adds non-uniform knots.

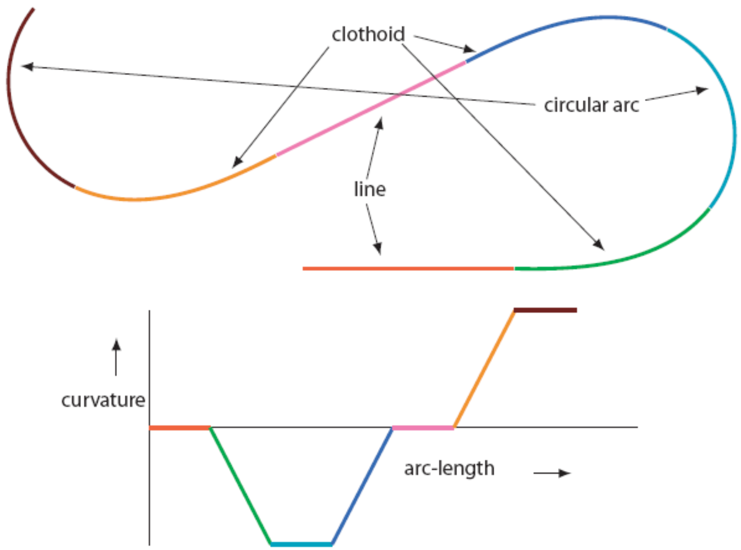
Curve Design Issues

- Continuity (smoothness, fairness and neatness).
- Control (local vs. global).
- Interpolation vs. approximation of constraints.
- Other geometric properties (planarity, invariance).
- Efficient analytic representation.

Smooth curves

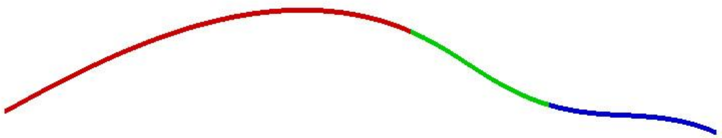
- **Fairness**: "*curvature continuous curves with a small number of segments of almost piecewise linear curvature*" [Farin et al. 87].

- Lines, circles and clothoids are the simplest primitives in curvature space.



Clothoids

Sketch curves often represent gestural information or capture design intent where the overall stroke appearance (*fairness*) is more important than the precise input.



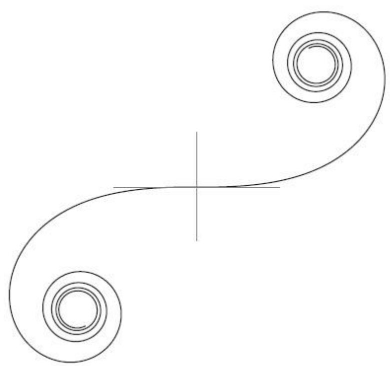
[**McCrae & Singh**, Sketching Piecewise Clothoid Curves, *SBIM 2008*]
http://www.dgp.toronto.edu/~mccrae/clothoid/

What are Clothoids?

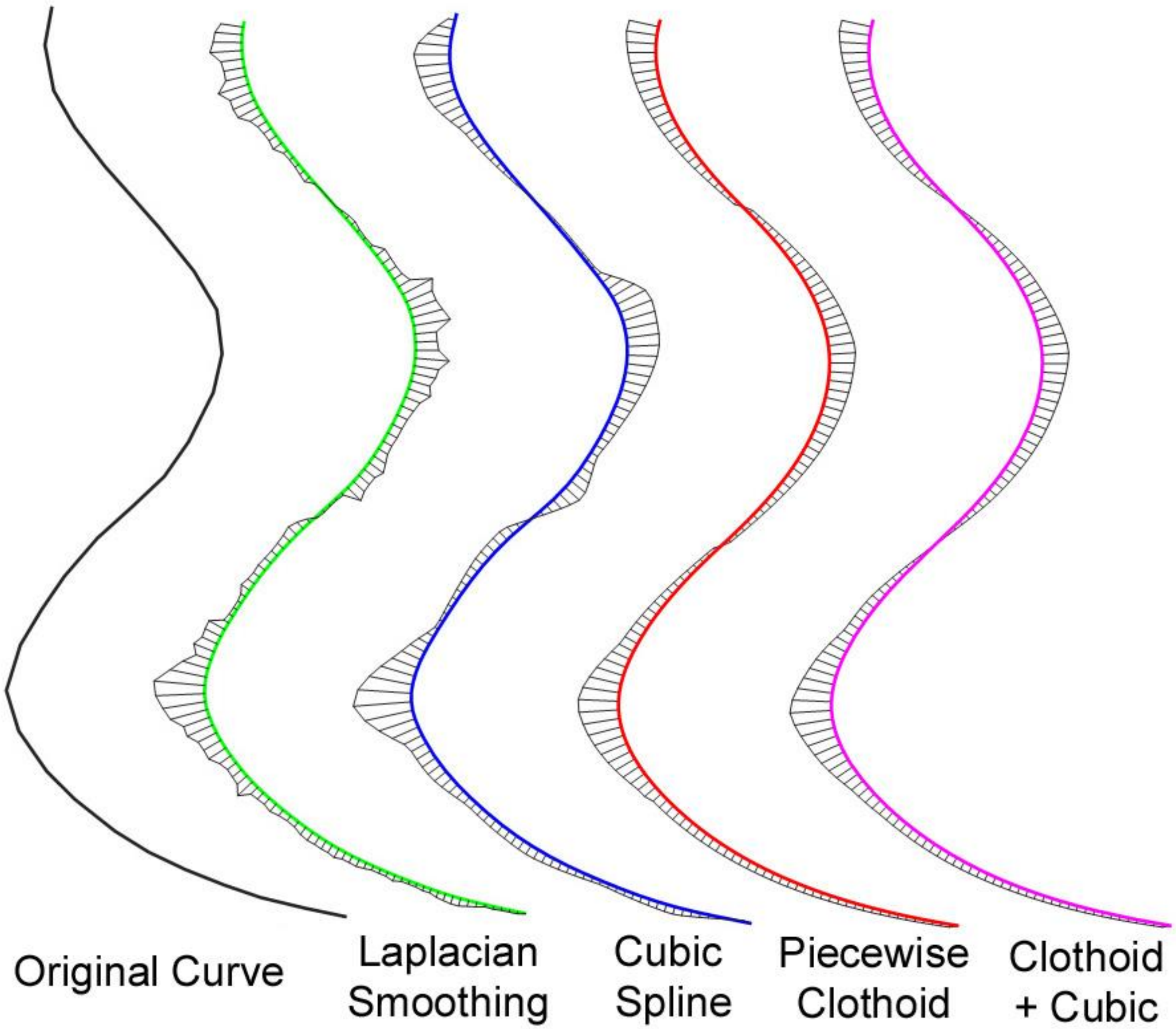
- Curves whose curvature changes linearly with arc-length.
- Described by Euler in 1774, a.k.a. Euler spiral.
- Studied in diffraction physics, transportation engineering (constant lateral acceleration) and robot vehicle design (linear steering).

$$C(t) = \int_0^t \cos \frac{\pi}{2} u^2 du$$

$$S(t) = \int_0^t \sin \frac{\pi}{2} u^2 du$$



Comparative approaches to fairing

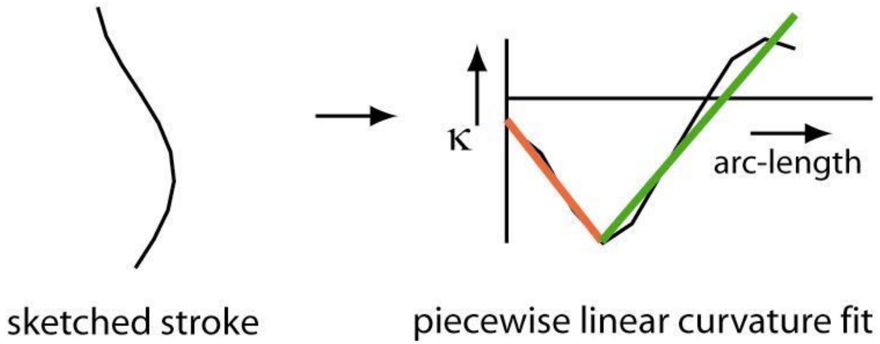


Approach

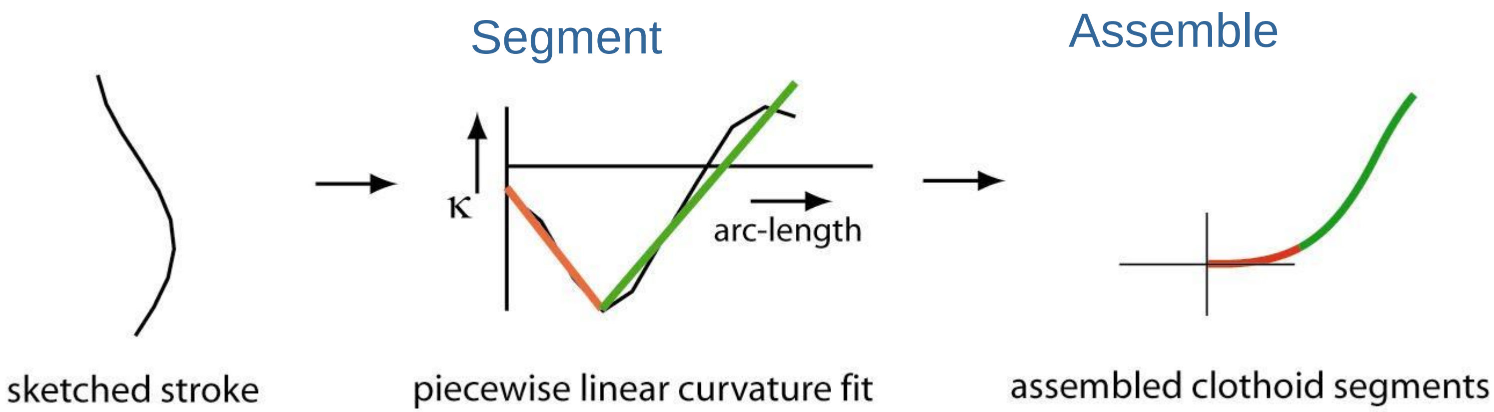
sketched stroke

Approach

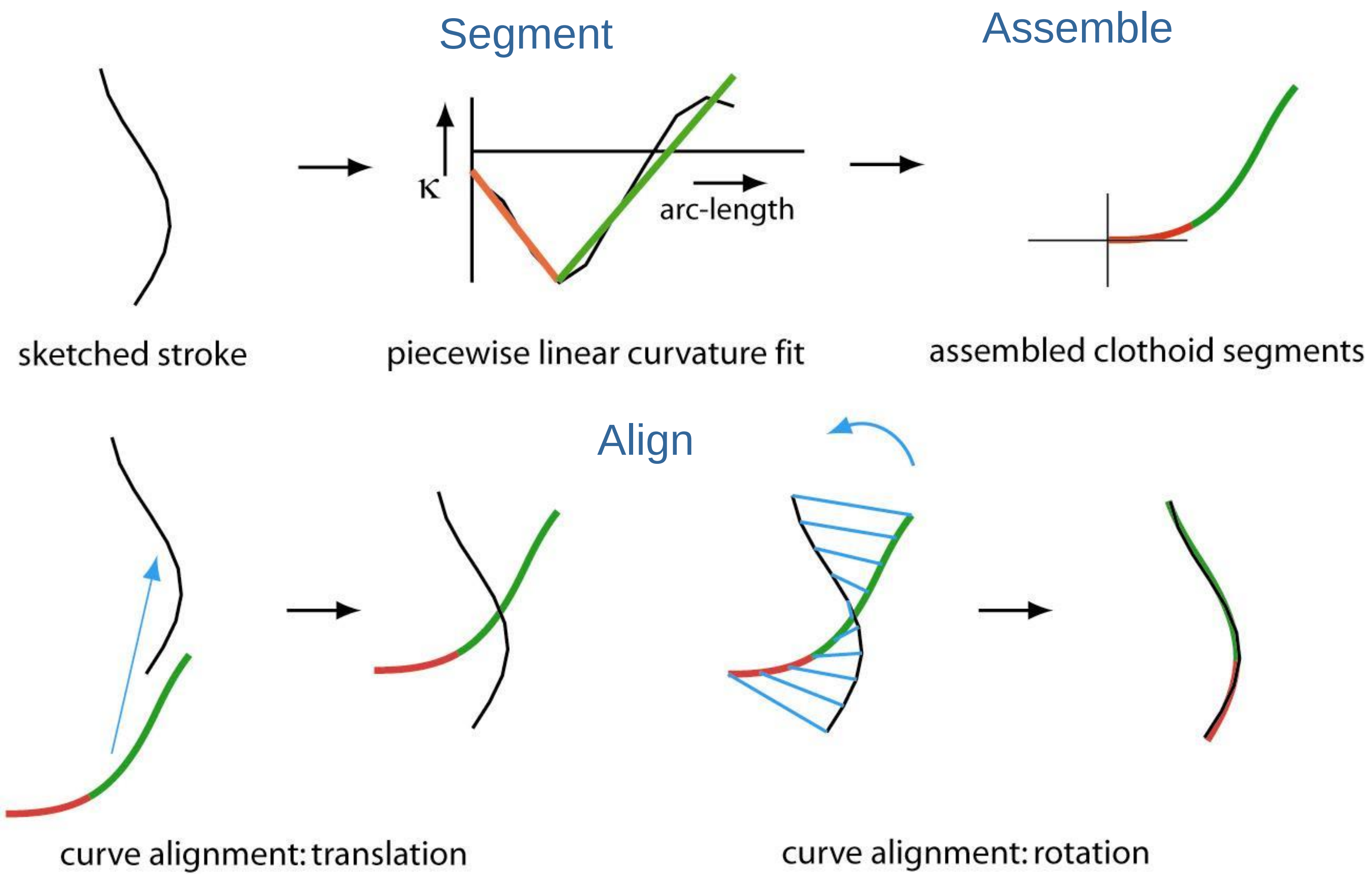
Segment



Approach

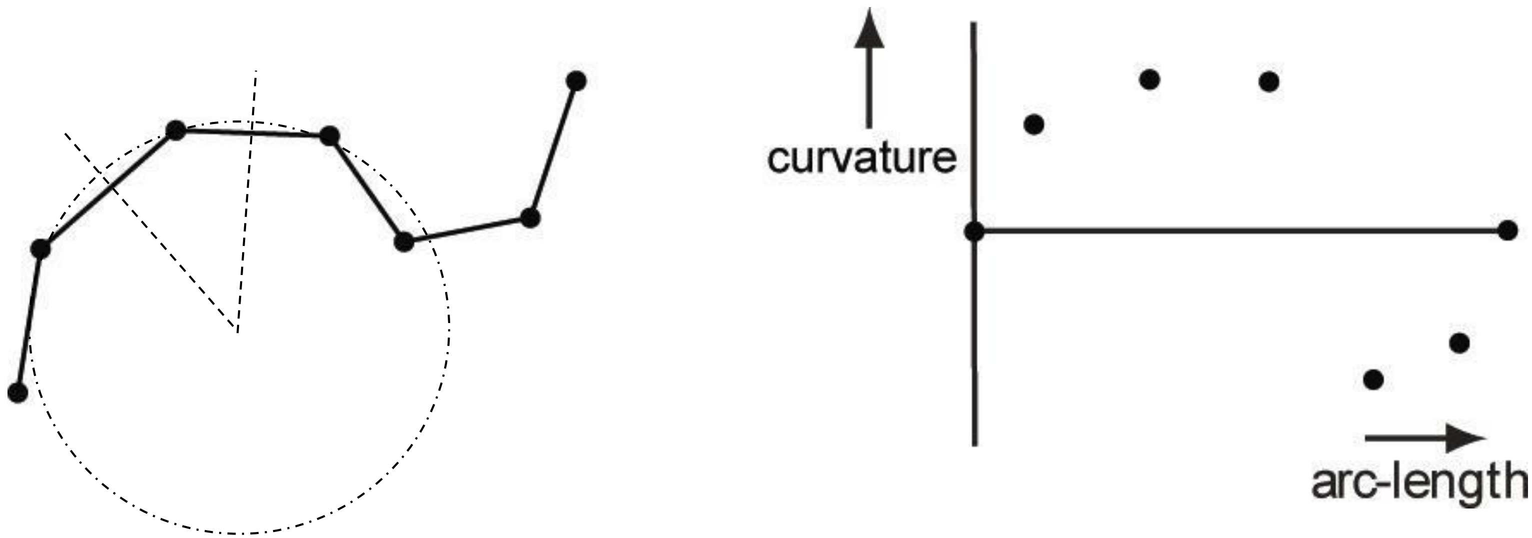

Segment
Assemble
κ
arc-length
sketched stroke
piecewise linear curvature fit
assembled clothoid segments

Approach


Segment
Assemble
κ
arc-length
sketched stroke
piecewise linear curvature fit
assembled clothoid segments
Align
curve alignment: translation
curve alignment: rotation

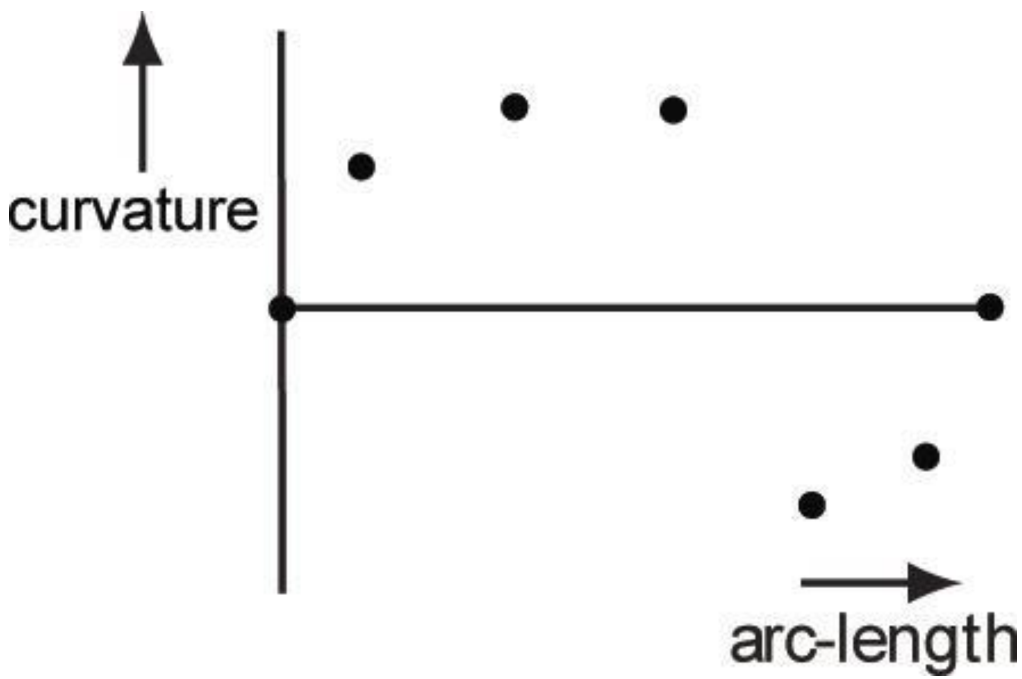
Piecewise Linear Curvature Fit

- Any discrete curvature estimator can be used to obtain curvature space points



Piecewise Linear Curvature Fit

- Find a ***small number*** of connected line segments that ***minimize fit error.***



Piecewise Linear Curvature Fit

- **Dynamic programming** (cost of fit matrix *M)*:

$$M(a,b) = \min_{a<k<b} \{M(a,k) + M(k,b), E_{fit}(a,b) + E_{cost}\}$$

$E_{fit}(a,b)$ is the fitting error of a line to points *a..b.*
E_{cost} is the penalty incurred to increment the number of line segments.

Piecewise Linear Curvature Fit

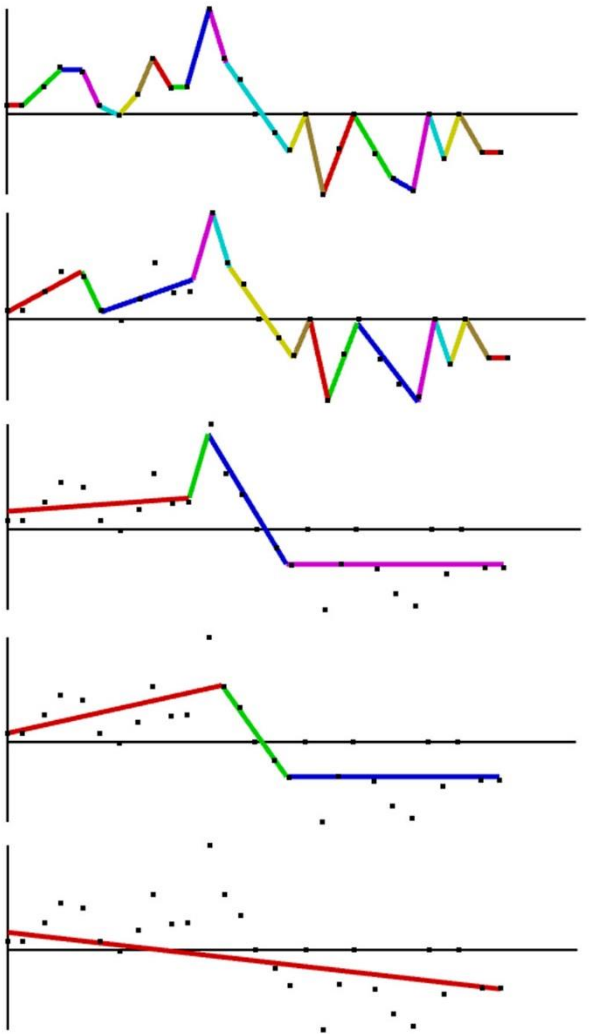
$E_{cost} = 0.01$

$E_{cost} = 0.02$

$E_{cost} = 0.035$

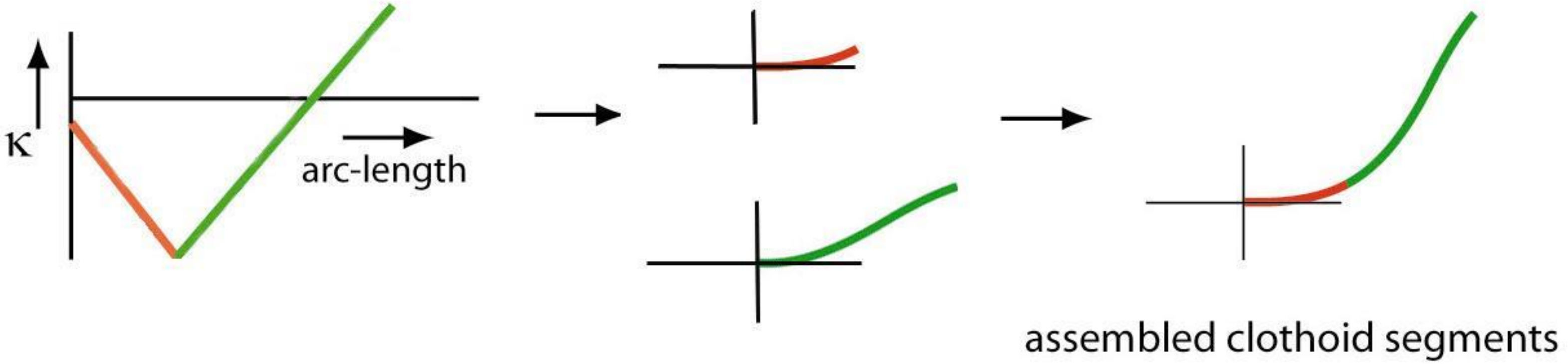
$E_{cost} = 0.05$

$E_{cost} = 0.08$



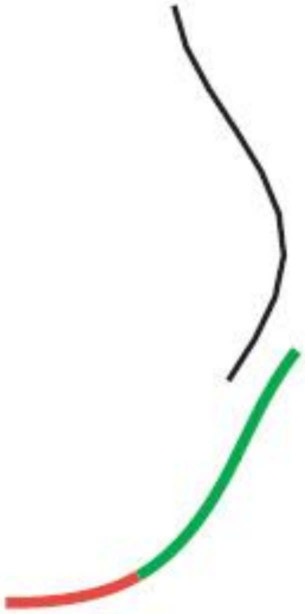
Assembly

- To assemble piecewise clothoid curve:
  - Map each curvature space line segment to a unique line, circle or clothoid curve segment.
  - Attach segments so they are position/tangent continuous
- Resulting curve has G^2 continuity



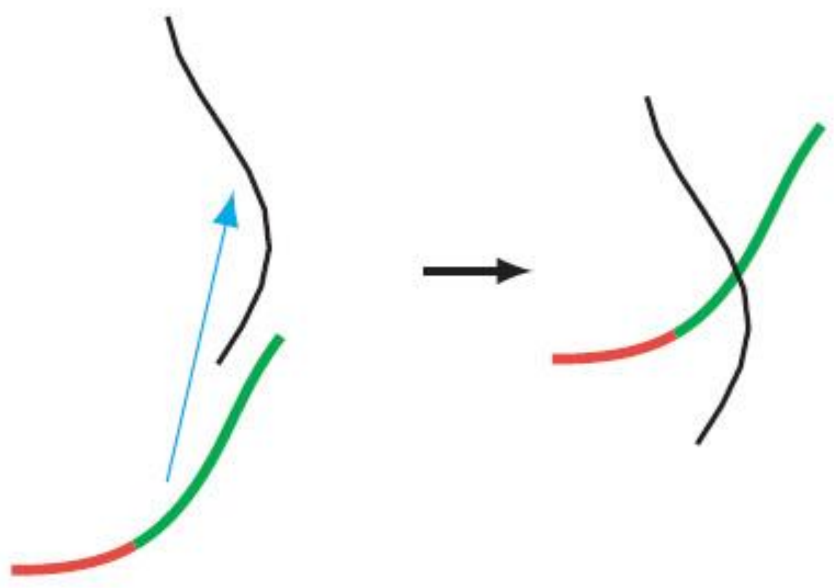
Align

Find a rigid transform that minimizes the sum of squared distance between arc-length corresponding points on the input polyline and piecewise clothoid curve. [Horn 1987]



Curve Alignment: Translation

- Translation is difference between the centroids of the points along both curves



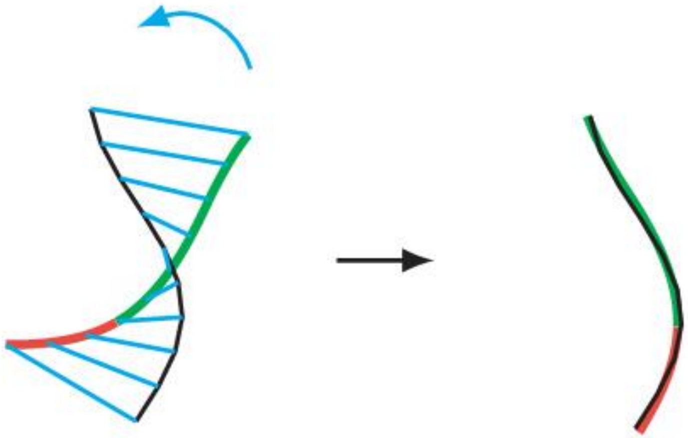
curve alignment: translation

Curve Alignment: Rotation

- Rotation minimizes weighted squared distances:
- Optimal *A* given by: $\sum_{i=0}^{n-1} w_i(Aq_i - p_i)^2$
- Rotation *R* extracted as

$$A = (\sum_{i=0}^{n-1} w_i p_i q_i^T)(\sum_{i=0}^{n-1} w_i q_i q_i^T)^{-1} = A_{pq}A_{qq}$$,where

$$R = A_{pq}S^{-1} \qquad S = \sqrt{A_{pq}^T A_{pq}}$$



curve alignment: rotation

Case Studies

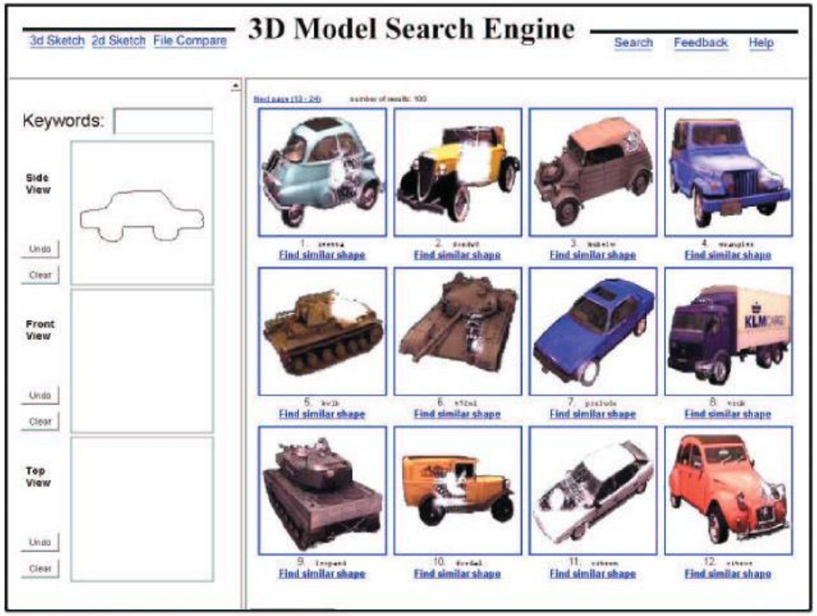
- Drive
- ILoveSketch.
- Teddy, Fibermesh.
- MeshMixer.

Model creation categories

- Suggestive systems
  - Input compared to template objects
  - *symbolic* or *visual memory*
- Constructive systems
  - Input directly used to create object
  - *perceptual* or *visual rules*

Suggestive systems

- User draws complete or gestural sketch.
- Sketch matched against object database or known primitives.



Funkhouser et al., *A Search Engine for 3D Models*, Proc. of SIGGRAPH'03, 2003.

Suggestive systems (matching 2D to 3D)

- Extract several contours for each object.
- Create feature vector
  - Direct comparison, eg. Euclidean distance.

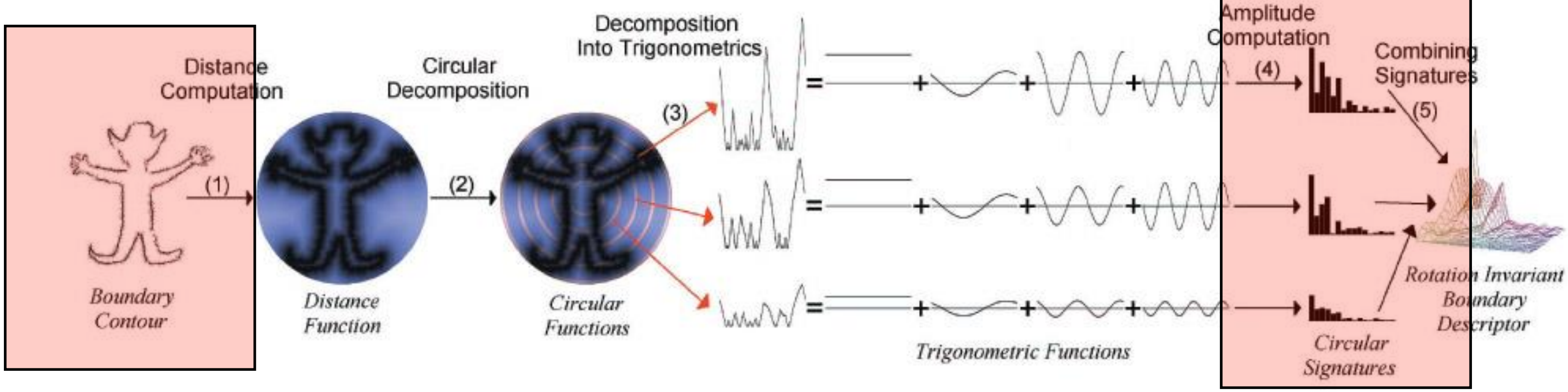
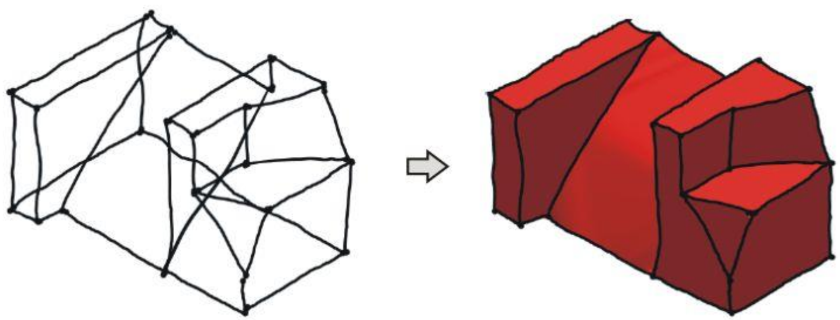


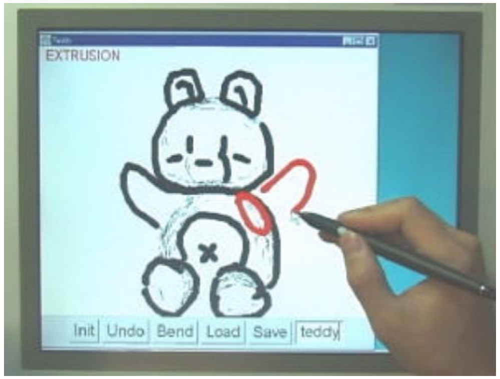
Fig. 9. Computing our shape descriptor for boundary contours.

Funkhouser et al., *A Search Engine for 3D Models*, Proc. of SIGGRAPH'03, 2003.

Constructive systems

- Rules and constraints rather than templates:
  - Restricting application domain (eg. sketching roads).
  - Restricting object type (eg. mechanical or organic).
  - Restricting task (eg. smoothing, cutting or joining).





M. Masry and H. Lipson, *A Sketch-Based Interface for Iterative Design and Analysis of 3D Objects*, EG SBIM'05, 2005.
T. Igarashi et al., ***Teddy**: A Sketching Interface for 3D Freeform Design*, Proc. of SIGGRAPH'99, 1999.

Drive: single-view sketching

A sketch-based system to create conceptual layouts of 3D path networks.

Drive features

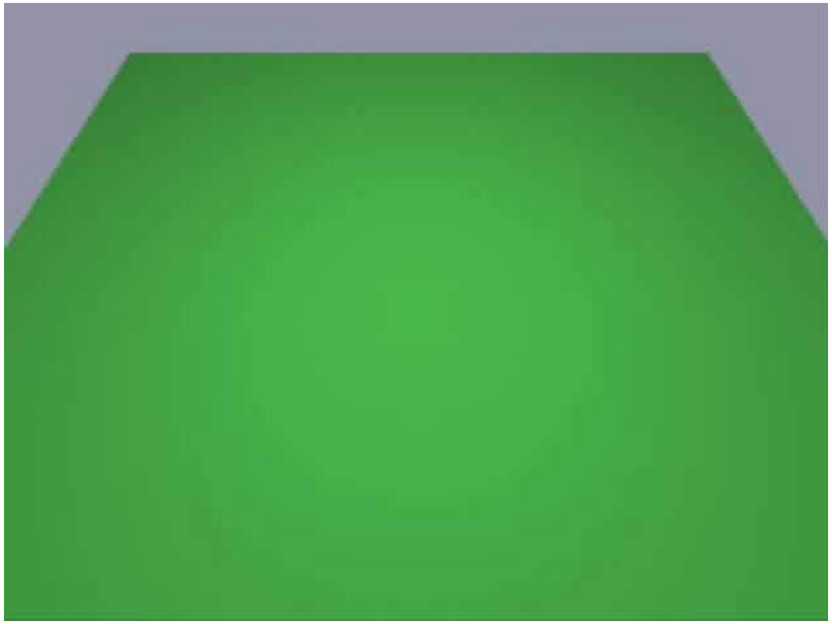
- Elegant interface:

  open stroke = path

  closed stroke = selection-action menu.
- Piecewise clothoid path construction.
- Crossing paths.
- Break-out lens. (single-view context)
- Terrain sensitive sketching.

[**McCrae & Singh**, Sketching based Path Design, *Graphics Interface 2009*]

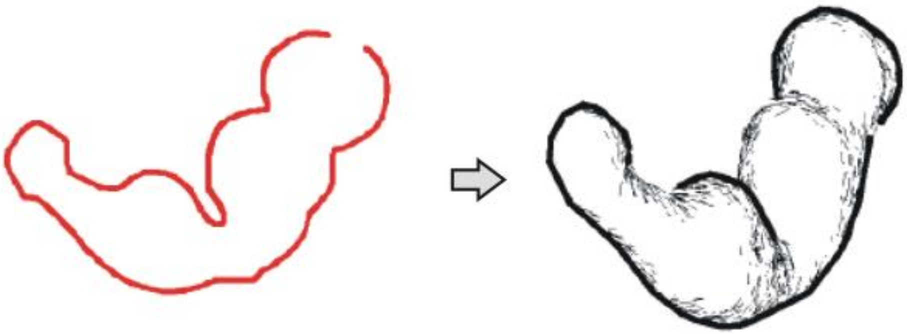
Drive



[**McCrae & Singh**, Sketching based Path Design, *Graphics Interface 2009*]

Teddy

- Teddy inflates a closed 2D stroke like blowing up a balloon.



T. Igarashi et al., ***Teddy**: A Sketching Interface for 3D Freeform Design*, Proc. of SIGGRAPH'99, 1999.

Inflation

- Offset surface proportionally to distance from spine of the contour
- Produces smooth blobby objects

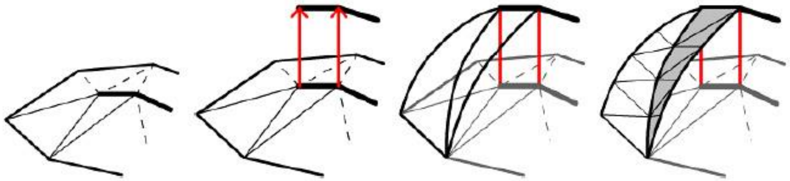
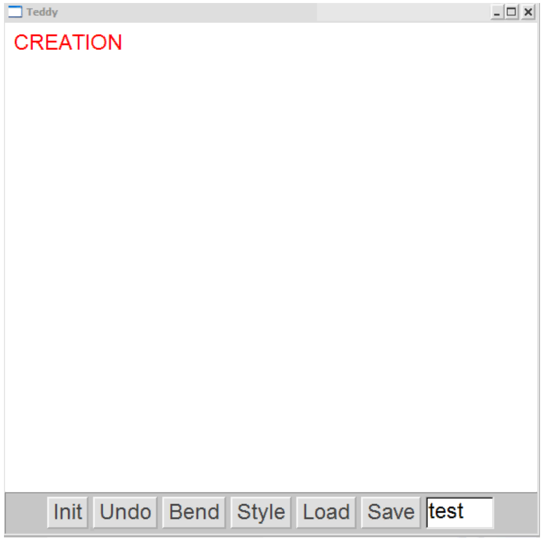


Figure 15: Polygonal mesh construction.



Igarashi et al., ***Teddy**: A Sketching Interface for 3D Freeform Design*, SIGGRAPH'99, 1999.

Skeleton extraction

- Delaunay triangulation
- Chordal axis transform

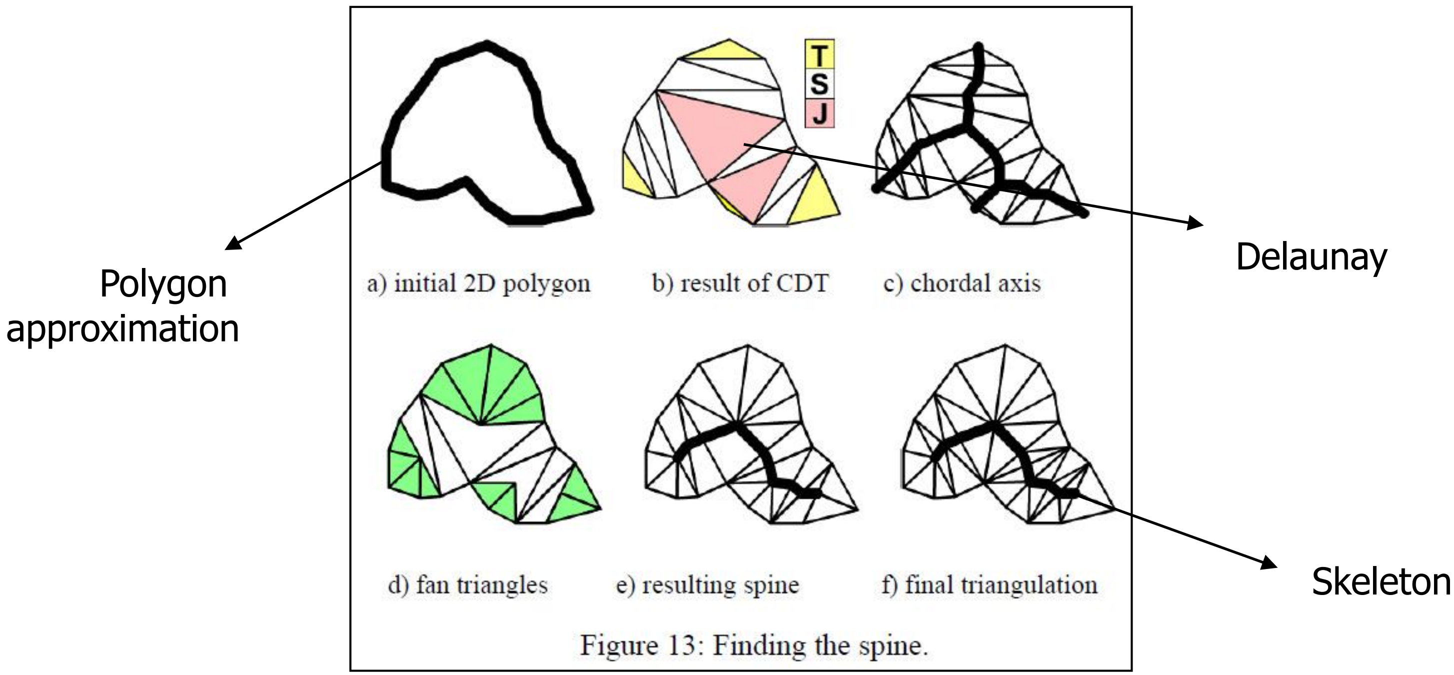
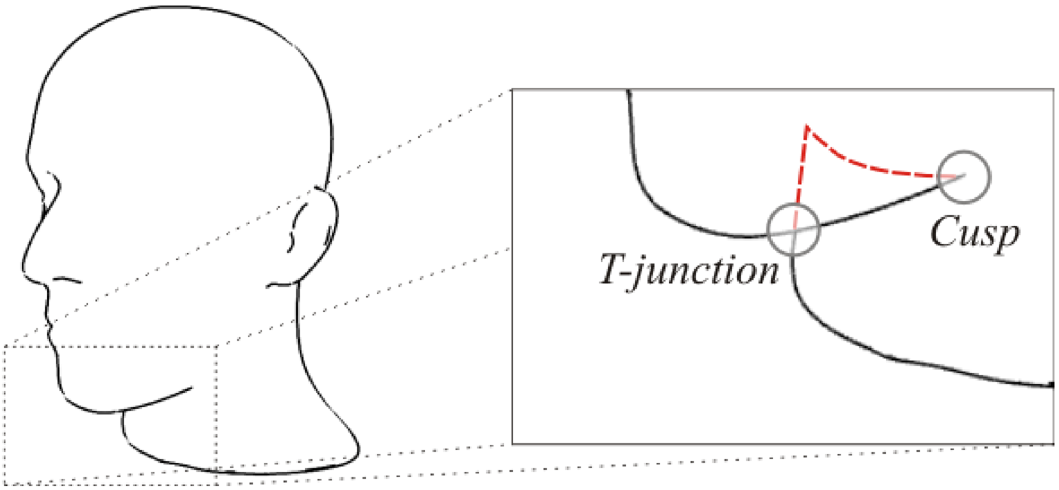


Figure 13: Finding the spine.

Igarashi et al., ***Teddy**: A Sketching Interface for 3D Freeform Design*, SIGGRAPH'99, 1999.

Trouble with contours and silhouettes

- Rarely planar.
- Can contain T-junctions and cusps.
- Occlusion.



3D Curve networks: surface optimization

- Surface results from solving non-linear system
  - 3D curves defines geometric constraints
  - Smoothness constraints

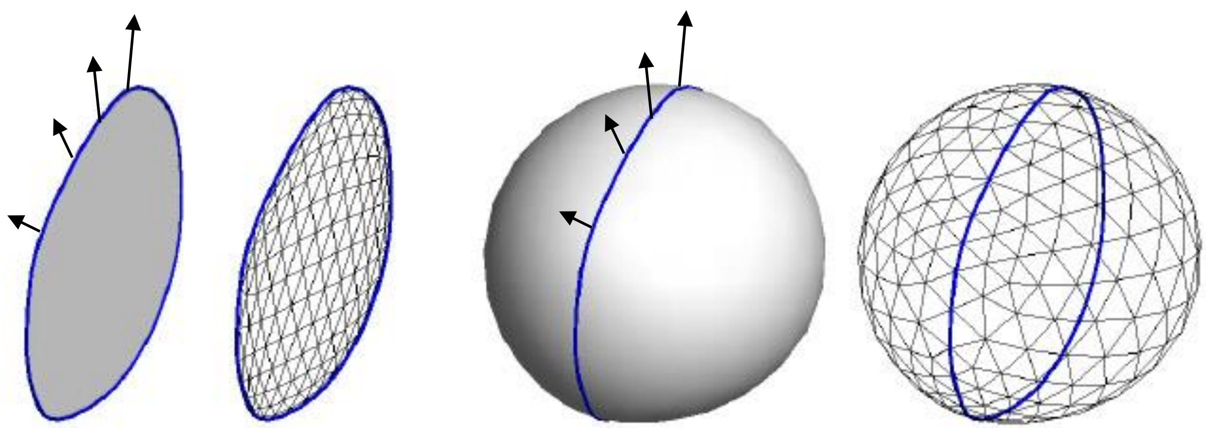
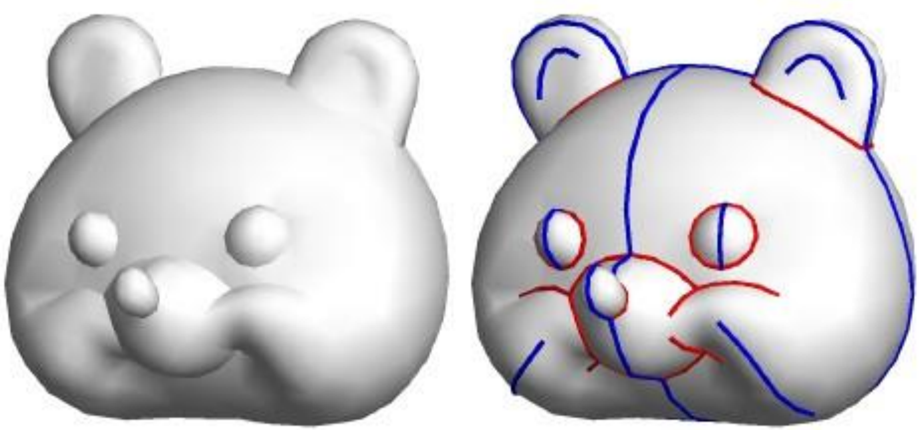


Figure 11: *The results of least-squares meshes (left) and our non-linear solution (right) for a planar curve.*

A. Nealen et al., ***FiberMesh****: Designing Freeform Surfaces with 3D Curves*, Proc. of SIGGRAPH'07, 2007.

FiberMesh

- User can specify additional curves on the surface
  - Further constraints that define the surface
  - Sharp features



A. Nealen et al., ***FiberMesh**: Designing Freeform Surfaces with 3D Curves*, Proc. of SIGGRAPH'07, 2007.

Case Studies

- Drive
- Teddy, Fibermesh.
- ILoveSketch.
- MeshMixer.

I ♥ SKETCH (multi-view sketching)

A corpus of research in sketch based modeling exists without a single such system in practical use…

Why?

- No clear overall user workflow.
- Insufficient vocabulary and quality of 3D curves.
- Poor transition from 2D sketching practice.

I♥SKETCH: multi-view sketching

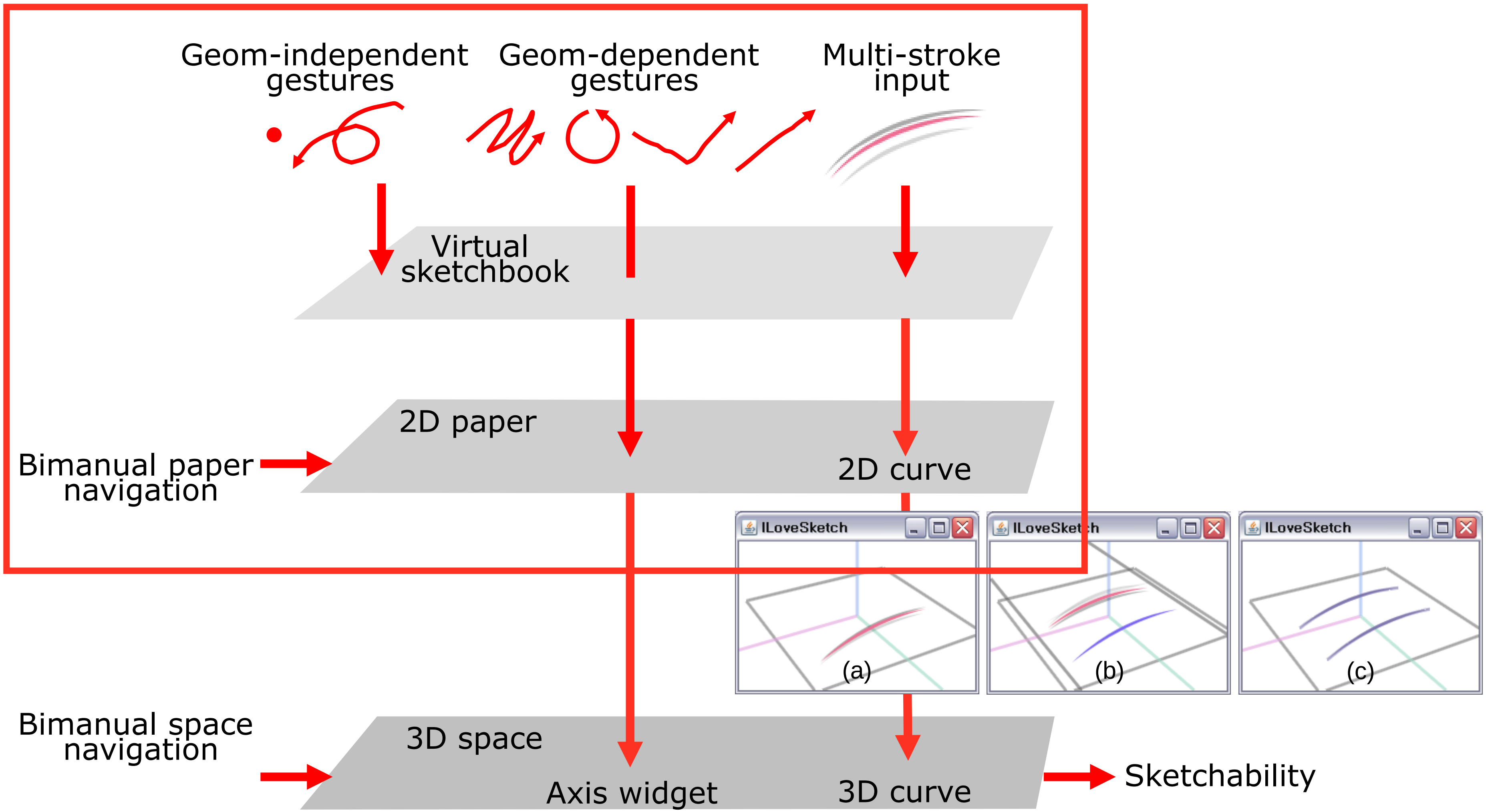
I♥SKETCH: multi-view sketching

A judicious leap from 2D to 3D.

- Presents a virtual 2D sketchbook with simple paper navigation and automatic rotation for ergonomic *pentimenti* style 2D sketching.
- Seamless transition to 3D with a suite of *multi-view curve sketching* tools with context switching based on *sketchability*.

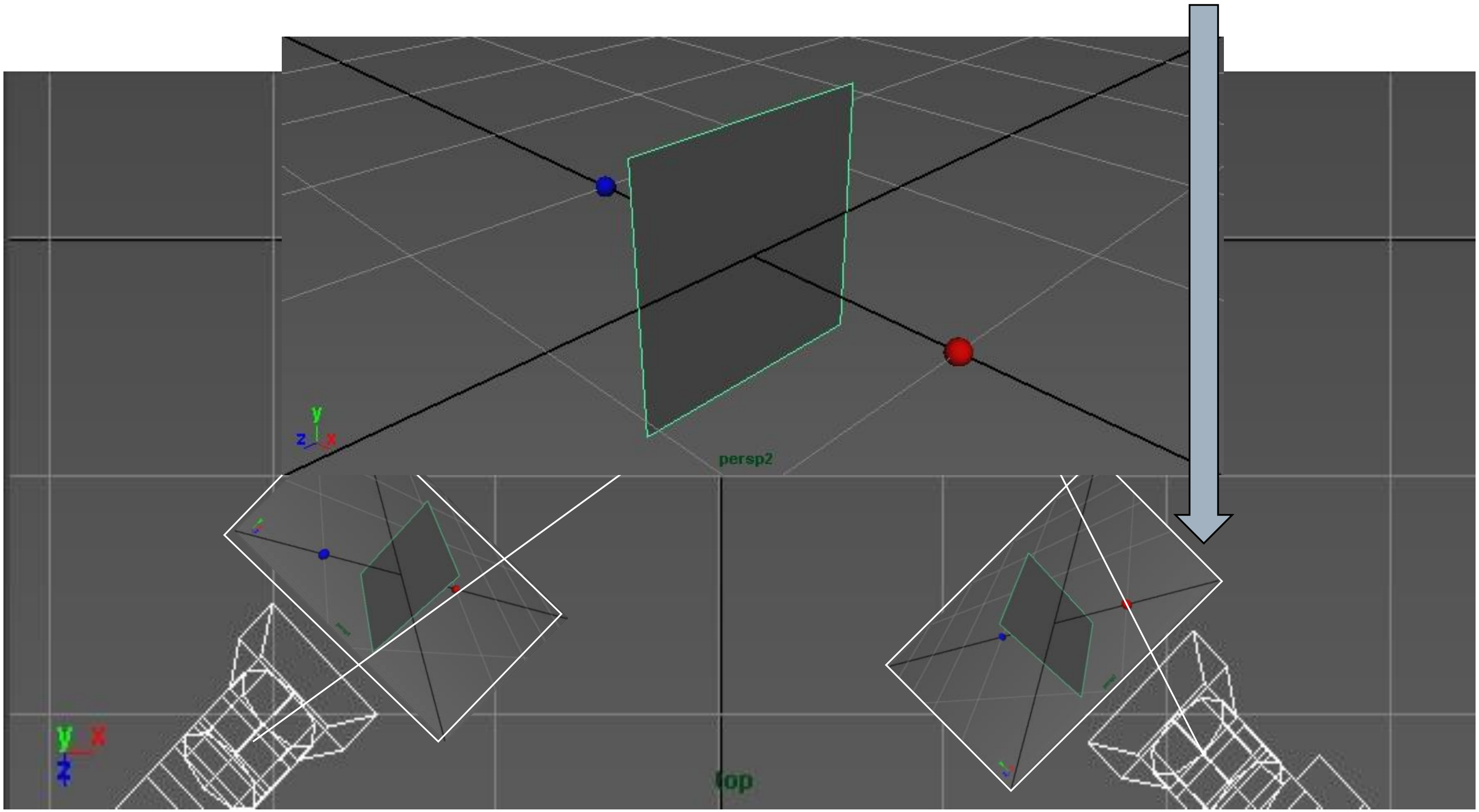
[**Bae, Balakrishnan & Singh**, ILoveSketch: As-natural-as-possible sketching system for creating 3D curve models. *UIST 2008*]
www.ilovesketch.com

I ♥ SKETCH



Geom-independent gestures
Geom-dependent gestures
Multi-stroke input
Virtual sketchbook
2D paper
2D curve
Bimanual paper navigation
ILoveSketch
(a)
(b)
(c)
Bimanual space navigation
3D space
Axis widget
3D curve
Sketchability

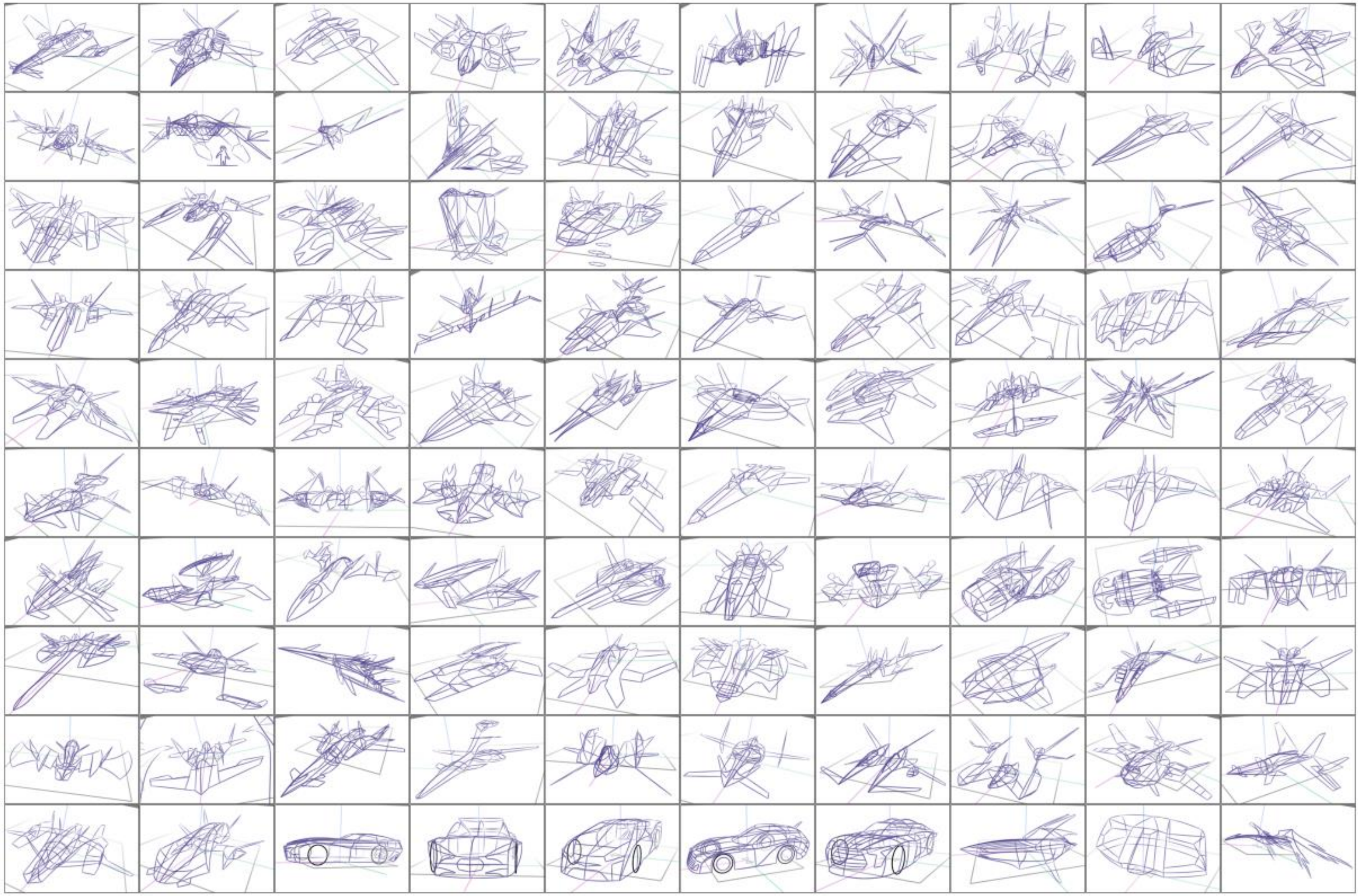
I ♥ SKETCH : epi-polar symmetry



I ♥ **SKETCH** (at SIGGRAPH 09 eTech)

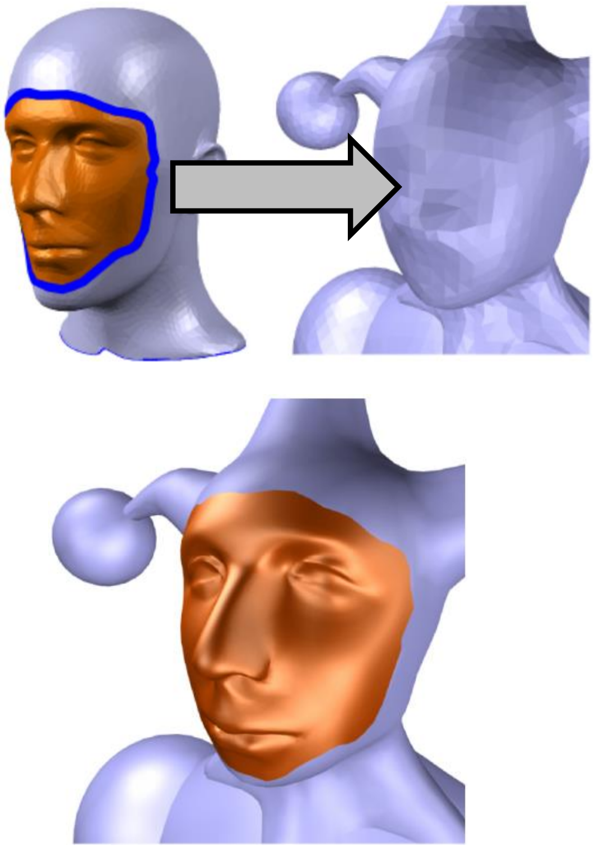
100 models created over 4 days (made public for research)

http://www.dgp.toronto.edu/~shbae/ilovesketch_siggraph2009.htm

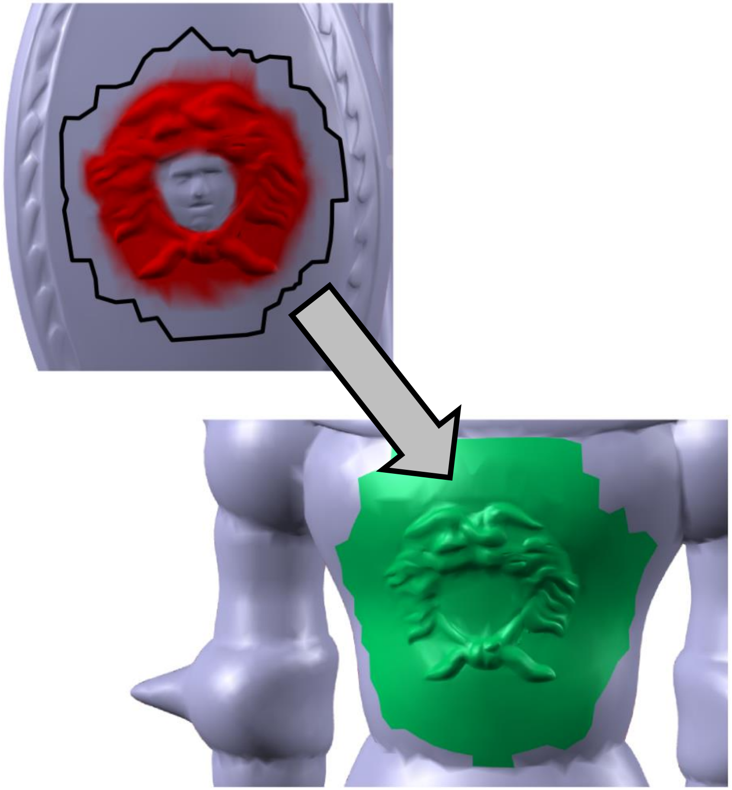


Meshmixer: 3D model composition

Composing **Parts**:
Mesh Drag-and-Drop

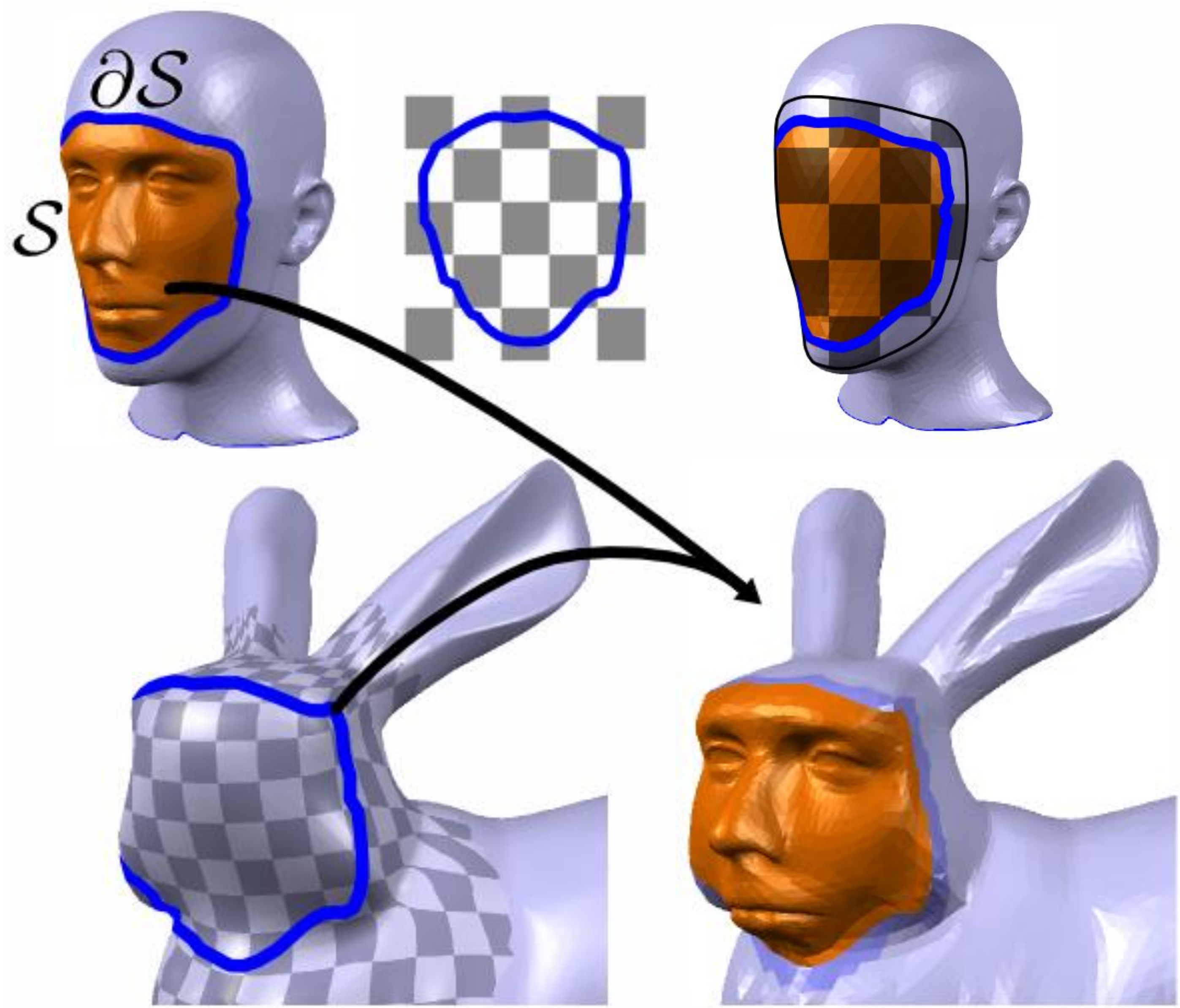


Composing **Details**:
Mesh Clone Brush



MeshMixer

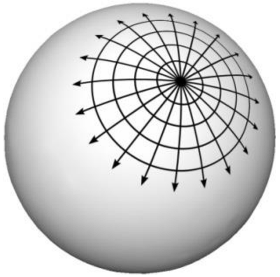
Parametric boundary based deformation

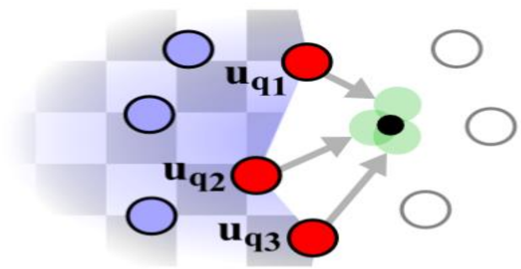


Two main tasks

1. Parameterize Local Region

   Improved Discrete Exponential Map

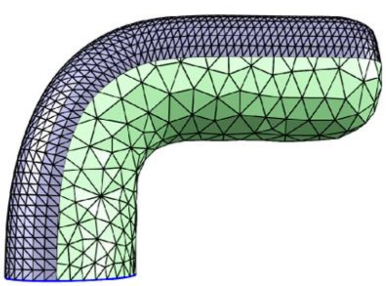


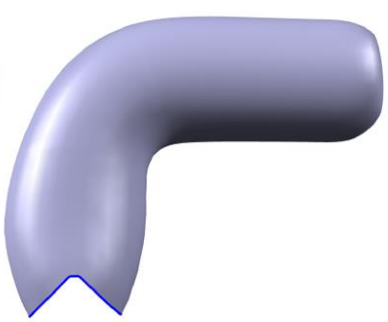


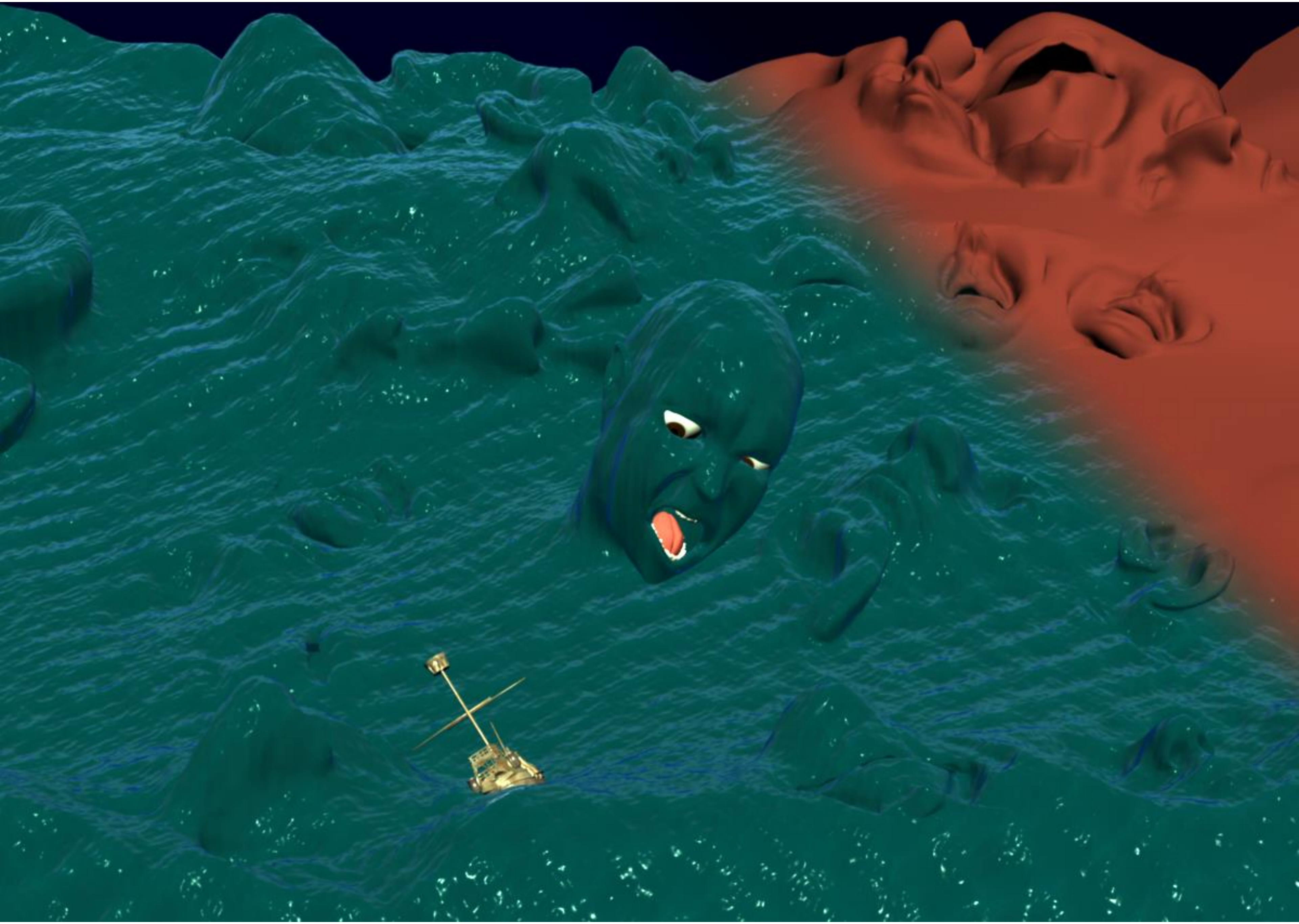
Our "Upwind-Average" DEM

2. Deform Part based on boundary

   COILS geometric deformer







Next Lecture…

Meshes, Discrete Differential Geometry, features.