Topic 3:

2D Transformations

- Simple Transformations
- Homogeneous coordinates
- Homogeneous 2D transformations
- Affine transformations & restrictions

Transformations

Transformation/Deformation in Graphics:

A function f, mapping points to points.
simple transformations are usually invertible.

$$[x\ y]^T \underset{f^{-1}}{\overset{f}{\rightleftarrows}} [x'\ y']^T$$

Applications:

- Placing objects in a scene.
- Composing an object from parts.
- Animating objects.

Processing Tree Demo!
https://processing.org/examples/tree.html

Lets start out simple...

Translate a point $[x\ y]^T$ by $[t_x\ t_y]^T$:

$$x' = x + t_x$$
$$y' = y + t_y$$

Rotate a point $[x\ y]^T$ by an angle t :

$$x' = x \cos t - y \sin t$$
$$y' = x \sin t + y \cos t$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$x' = x\, s_x$$
$$y' = y\, s_y$$

Representing 2D transforms as a 2x2 matrix

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Translate?

Cartesian ⇔ Homogeneous 2D Points

Cartesian $[x\ y]^T$ => Homogeneous $[x\ y\ 1]^T$

Homogeneous $[x\ y\ w]^T$ => Cartesian $[x/w \quad y/w \quad 1]^T$

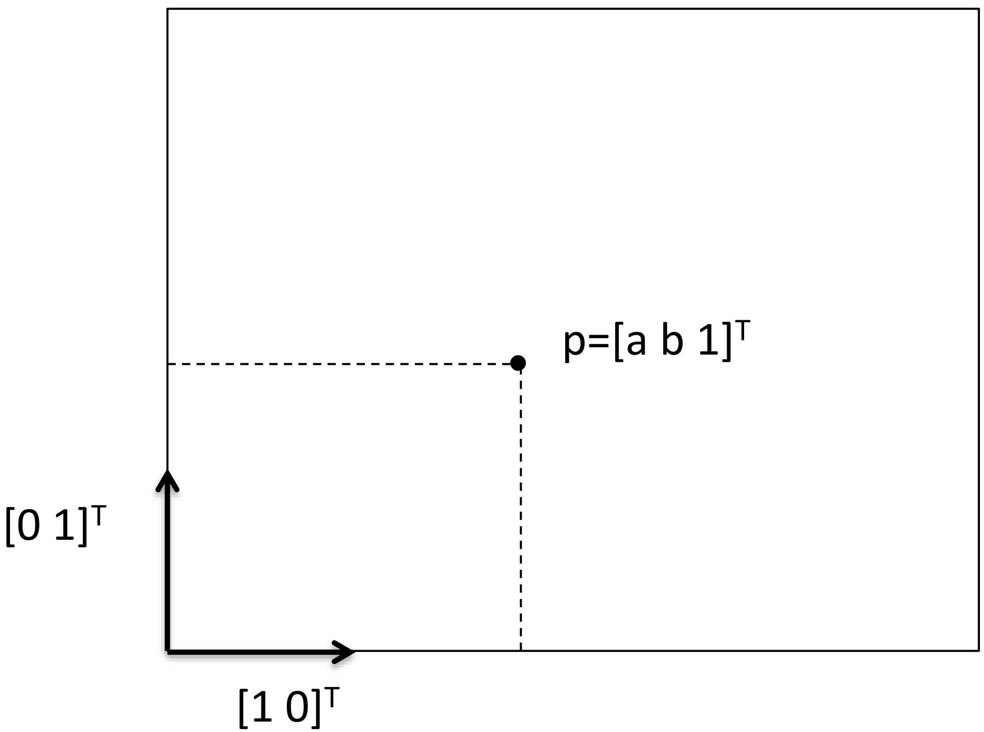
Homogeneous points are equal if they represent the same Cartesian point. For eg. $[4\ {-6}\ 2]^T = [-6\ 9\ {-3}]^T$.

What about w=0?

Points at ∞ in Homogeneous Coordinates

$[x\ y\ w]^T$ with w=0 represent points at infinity, though with direction $[x\ y]^T$ and thus provide a natural representation for **vectors**, distinct from **points** in Homogeneous coordinates.

Points as Homogeneous 2D Point Coords



$$p = a*[1\ 0\ 0]^T + b*[0\ 1\ 0]^T + [0\ 0\ 1]^T$$

basis vectors   origin

Line Equations in Homogeneous Coordinates

A line given by the equation

$$ax+by+c=0$$

can be represented in Homogeneous coordinates as:

$l=[a\ b\ c]$, making the line equation

$$l.p= [a\ b\ c][x\ y\ 1]^T=0.$$

Aside: cross product as a matrix

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} [x\ y\ 1]^T$$

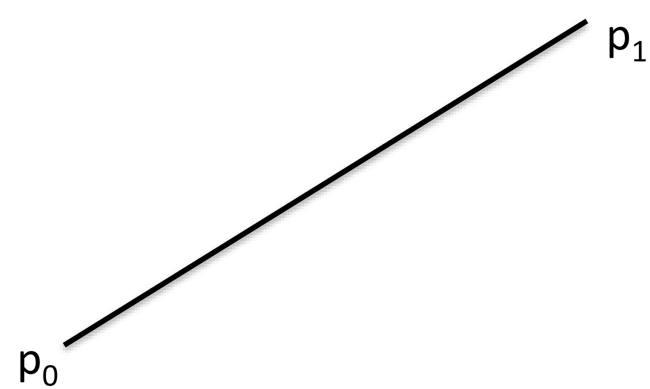
The Line Passing Through 2 Points

For a line l that passes through two points p_0, p_1

we have $l.p_0 = l.p_1 = 0$.

As vectors we can thus write l using a cross product as:

$l = p_0 \times p_1$



Remember *uxv* $= \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$

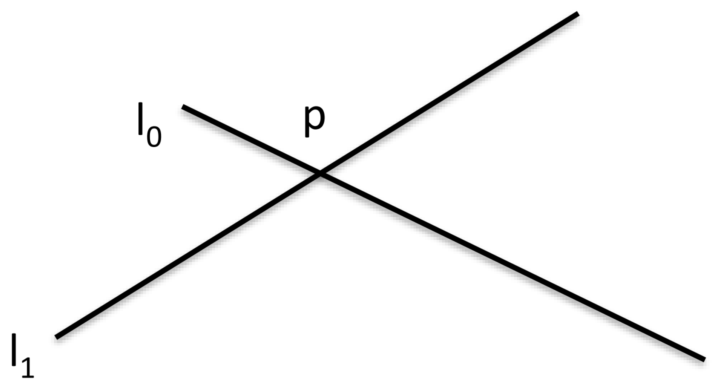
Point of intersection of 2 lines

For a point that is the intersection of two lines l_0, l_1

we have $p.l_0 = p.l_1 = 0$.

We can write p using a cross product as:

$p = l_0 \times l_1$



What happens when the lines are parallel?

Remember $uxv = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$

Representing 2D transforms as a 3x3 matrix

Translate a point $[x\ y]^T$ by $[t_x\ t_y]^T$:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotate a point $[x\ y]^T$ by an angle t :

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Scale a point $[x\ y]^T$ by a factor $[s_x\ s_y]^T$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Properties of 2D transforms

…these 3x3 transforms have a variety of properties. most generally they map **lines** to **lines**. Such invertible transforms are also called **Homographies**.

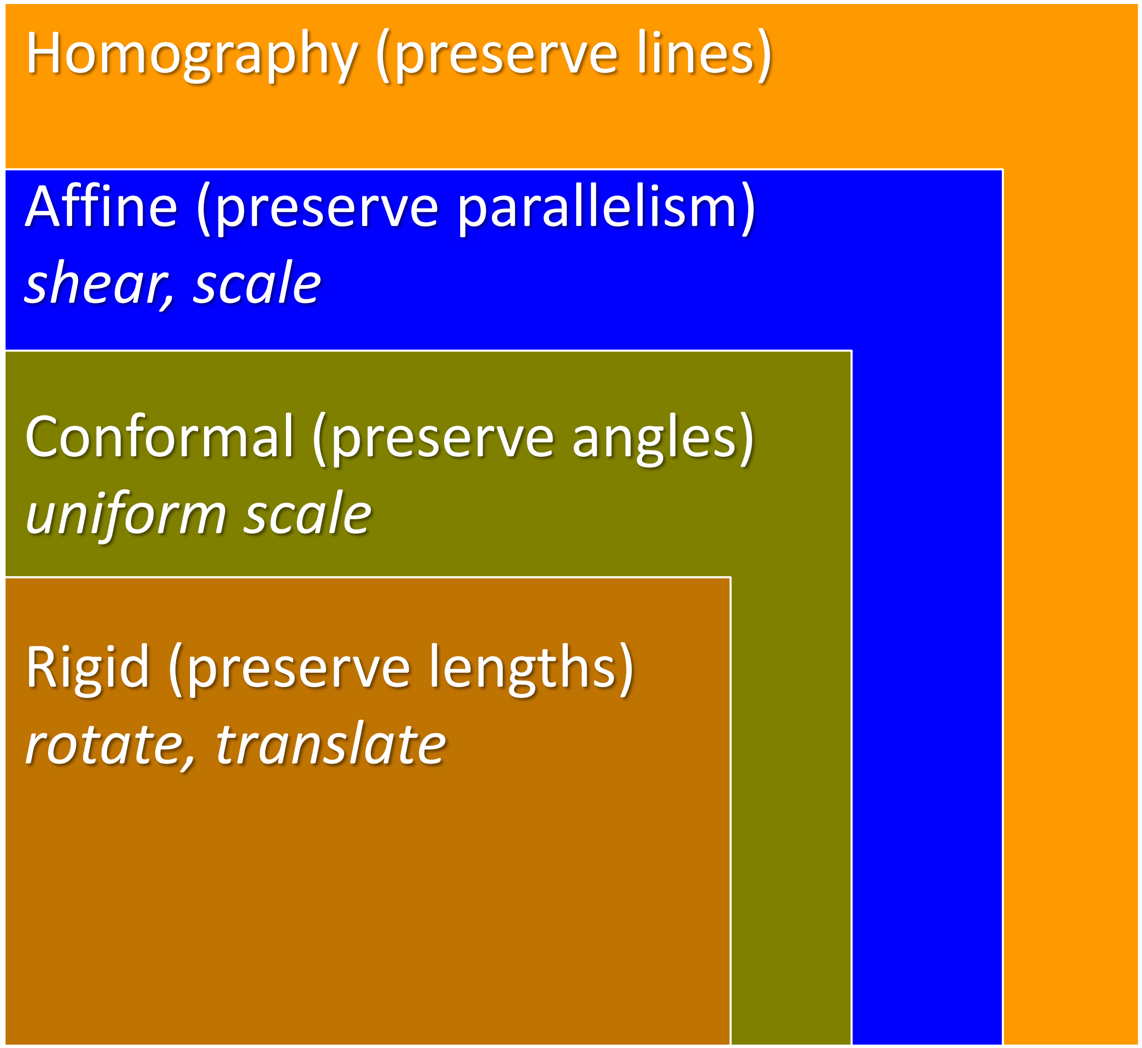
…a more restricted set of transformations also preserve parallelism in lines. These are called **Affine** transforms.

…transforms that further preserve the angle between lines are called **Conformal**.

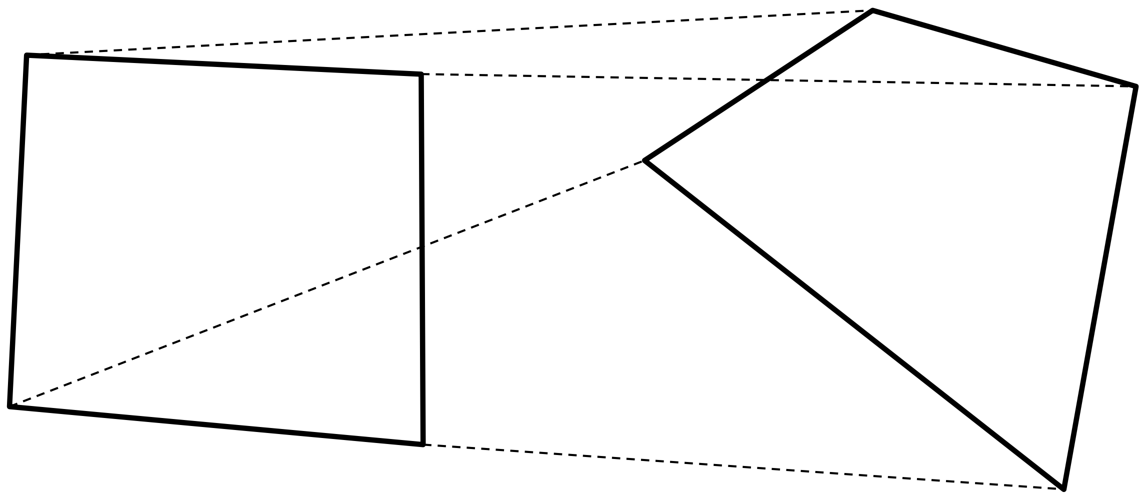
…transforms that additionally preserve the lengths of line segments are called **Rigid**.

Where do translate, rotate and scale fit into these?

Properties of 2D transforms



Homography: mapping four points



How does the mapping of 4 points uniquely define the 3x3 Homography matrix?

Homography: preserving lines

Show that if points p lie on some line l,
then their transformed points p' also lie on some line l'.

Proof:
We are given that $l.p = 0$ and $p'=Hp$. Since H is invertible, $p=H^{-1}p'$.
Thus $l.(H^{-1}p')=0 \Rightarrow (lH^{-1}).p'=0$, or p' lies on a line $l'= lH^{-1}$.

QED

Affine: preserving parallel lines

What restriction does the Affine property impose on H?

If two lines are parallel their intersection point at infinity, is of the form $[x\ y\ 0]^T$.

If these lines map to lines that are still parallel, then $[x\ y\ 0]^T$ transformed must continue to map to a point at infinity or $[x'\ y'\ 0]^T$

i.e. $$[x'\ y'\ 0]^T = \begin{pmatrix} * & * & * \\ * & * & * \\ ? & ? & ? \end{pmatrix} [x\ y\ 0]^T$$

Affine: preserving parallel lines

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i.e.

$$[x'\ y'\ 0]^T = \begin{pmatrix} \begin{pmatrix} & A & \\ & & \end{pmatrix} & \begin{bmatrix} t \\ \ \end{bmatrix} \\ 0 \quad 0 & 1 \end{pmatrix} [x\ y\ 0]^T$$

In Cartesian co-ordinates Affine transforms can be written as:

$$p' = Ap + t$$

Affine properties: composition

Affine transforms are closed under composition. i.e.
Applying transform (A_1,t_1) (A_2,t_2) in sequence results in an overall Affine transform.

$$p' = A_2 (A_1 p + t_1) + t_2 \quad \Rightarrow (A_2 A_1)p + (A_2 t_1 + t_2)$$

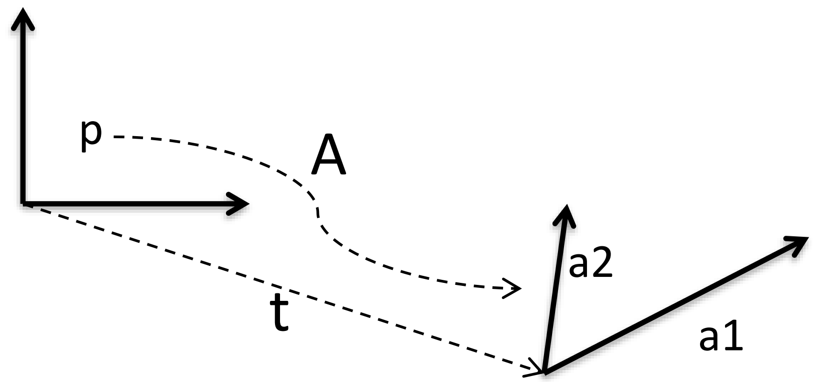
Affine properties: inverse

The inverse of an Affine transform is Affine.

- Prove it!

Affine transform: geometric interpretation

A change of basis vectors and translation of the origin

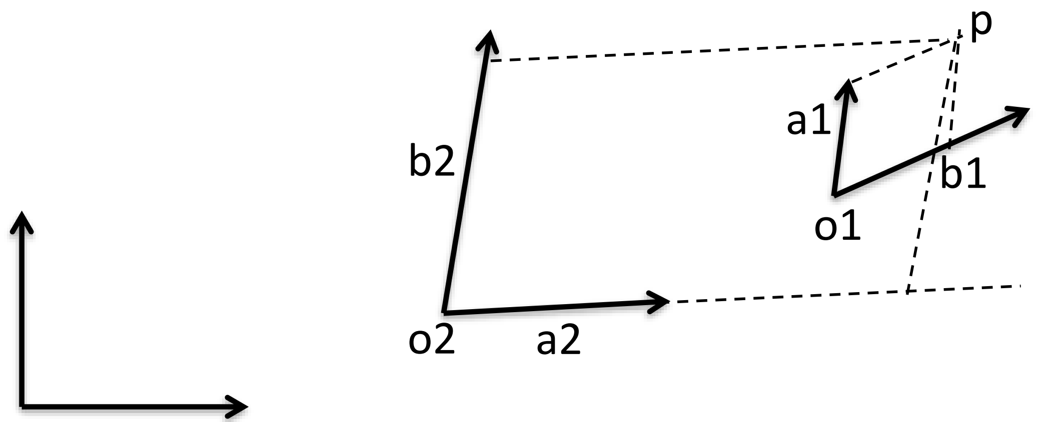


point p in the local coordinates of a reference frame defined by <a1,a2,t> is

$$\begin{pmatrix} \begin{pmatrix} a1 & a2 \end{pmatrix} & \begin{bmatrix} t \end{bmatrix} \\ 0 \quad 0 & 1 \end{pmatrix}^{-1} \begin{bmatrix} p \end{bmatrix}$$

Affine transform: change of reference frame

How can we transform a point p from one reference frame <a1,b1,o1>, to another frame <a2,b2,o2>?



Composing Transformations

Any sequence of linear transforms can be collapsed into a single 3x3 matrix by concatenating the transforms in the sequence.

In general transforms DO NOT commute, however certain combinations of transformations are commutative…

try out various combinations of translate, rotate, scale.

Rotation about a fixed point

The typical rotation matrix, rotates points about the origin. To rotate about specific point *q,* use the ability to compose transforms…

$$T_q \, R \, T_{-q}$$

Topic 4:

Coordinate-Free Geometry (CFG)

- A brief introduction & basic ideas

CFG: dimension free geometric reasoning

Points	p	[… 1]
Vectors	v	[… 0]
Lines	l	[…..]

Dot products, Cross products,
Length of vectors,
Weighted average of points…

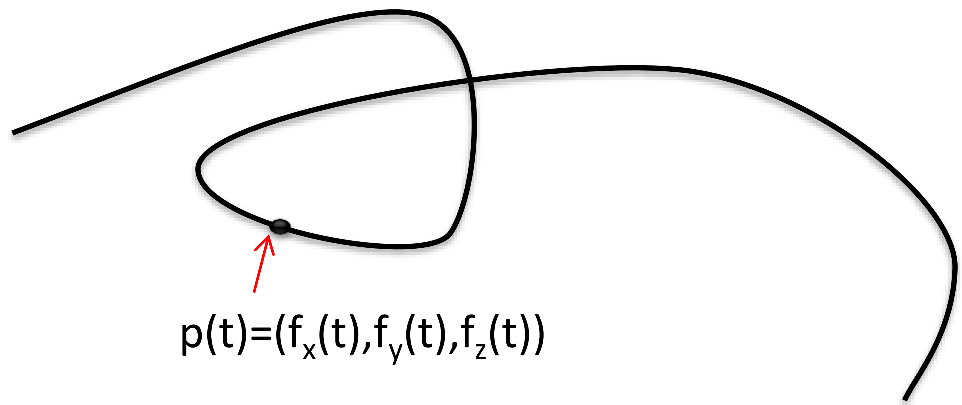
How do you find the angle between 2 vectors?

Topic 5:

3D Objects

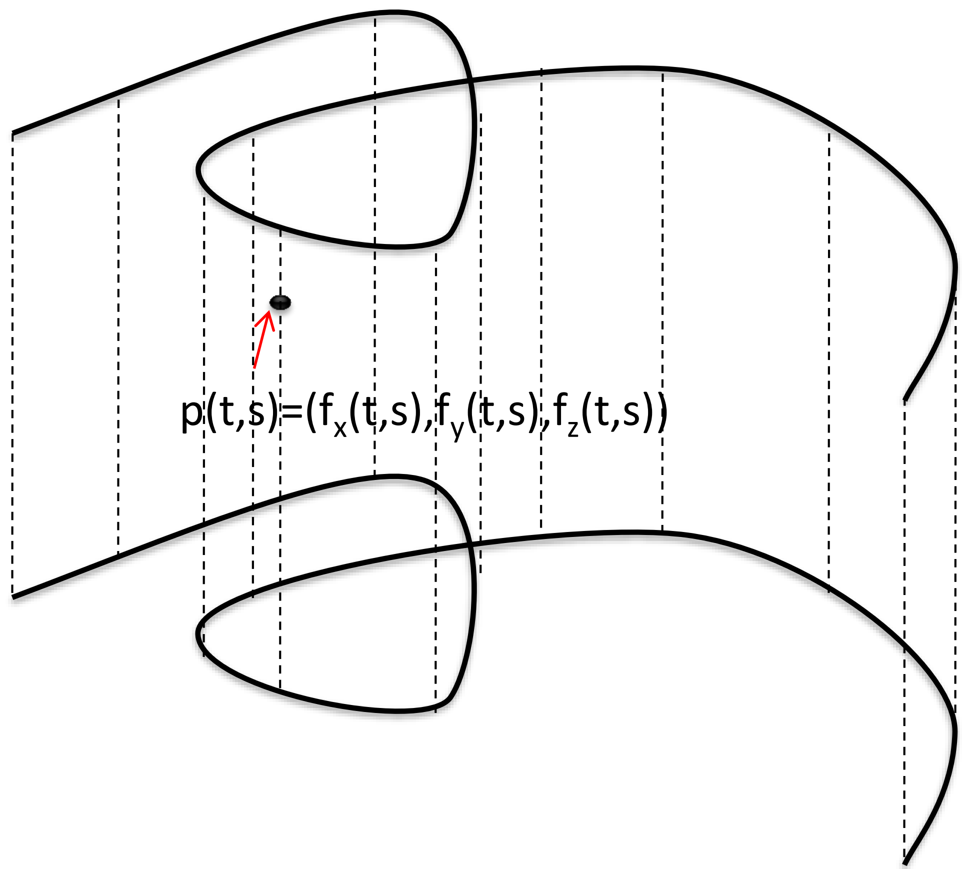
- General curves & surfaces in 3D
- Normal vectors, surface curves & tangent planes
- Implicit surface representations
- Example surfaces:
  surfaces of revolution, bilinear patches, quadrics

3D parametric curves

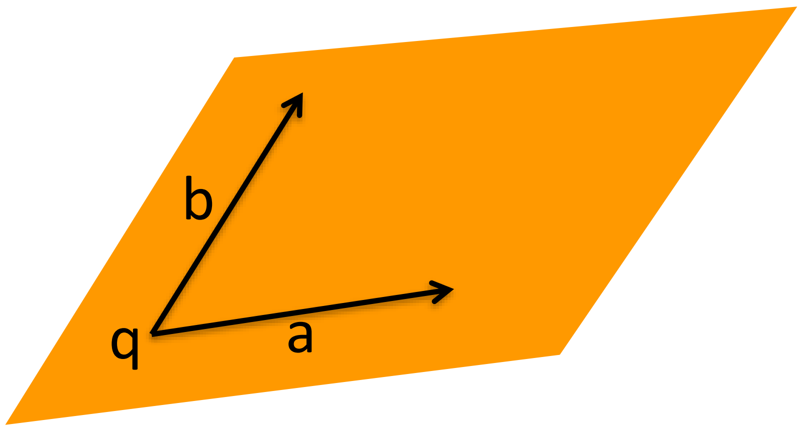


$p(t)=(f_x(t),f_y(t),f_z(t))$

3D parametric surfaces



3D parametric plane



p(s,t)= q + as +tb

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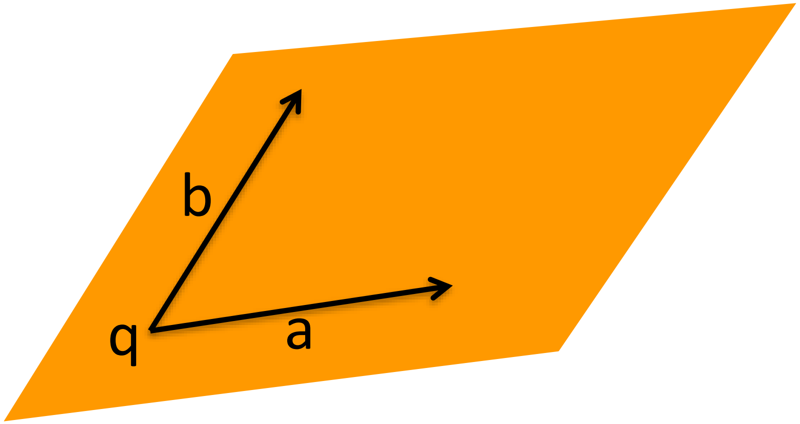
Tangent / Normal vectors of 2D curves

Explicit:	$y=f(x)$.	Tangent is dy/dx.
Parametric:	$x=f_x(t)$	Tangent is $(dx/dt, dy/dt)$
	$y=f_y(t)$	
Implicit:	$f(x,y) = 0$	Normal is gradient(f).
		direction of max. change

Given a tangent or normal vector in 2D how do we compute the other?

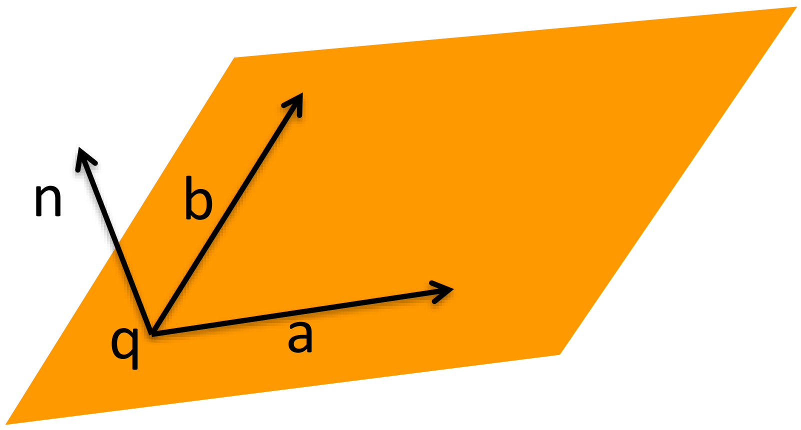
What about in 3D?

Normal vector of a plane



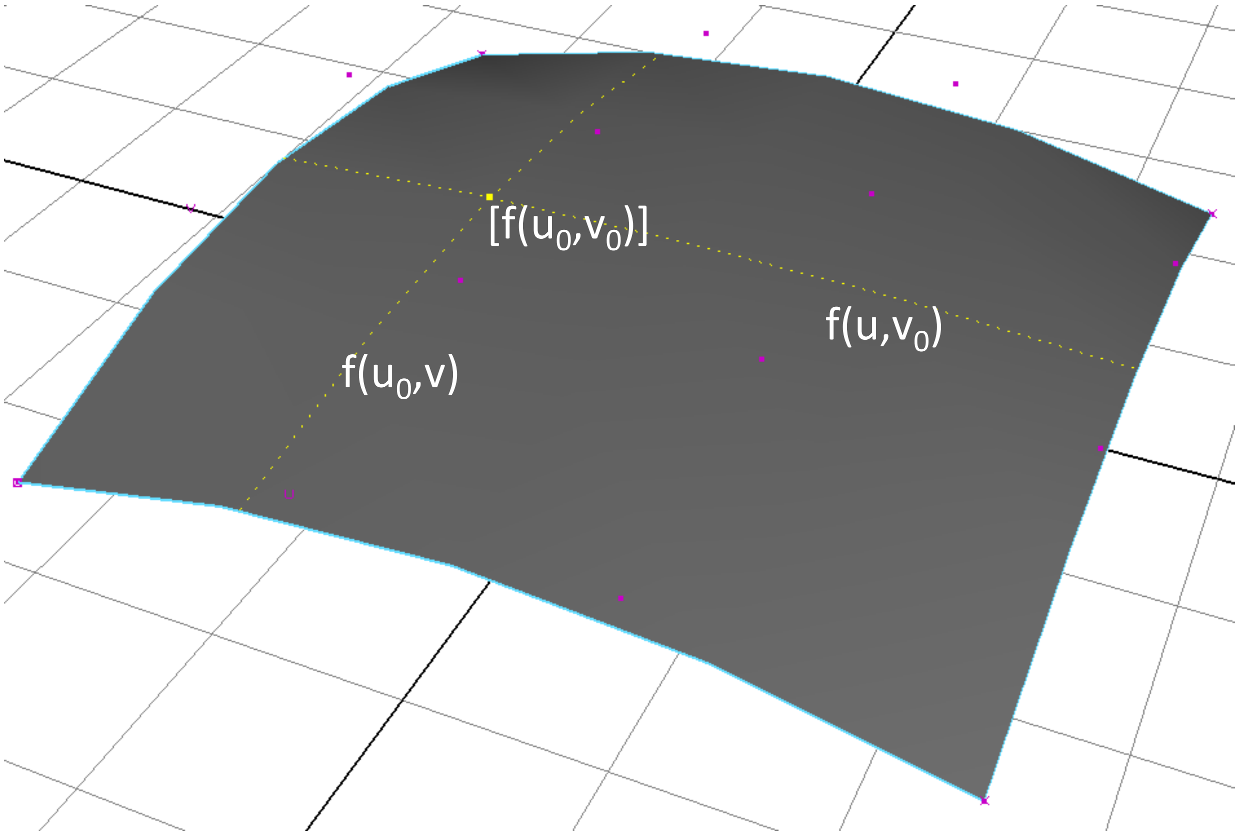
$p(s,t) = q + as + tb$

Normal vector of a plane

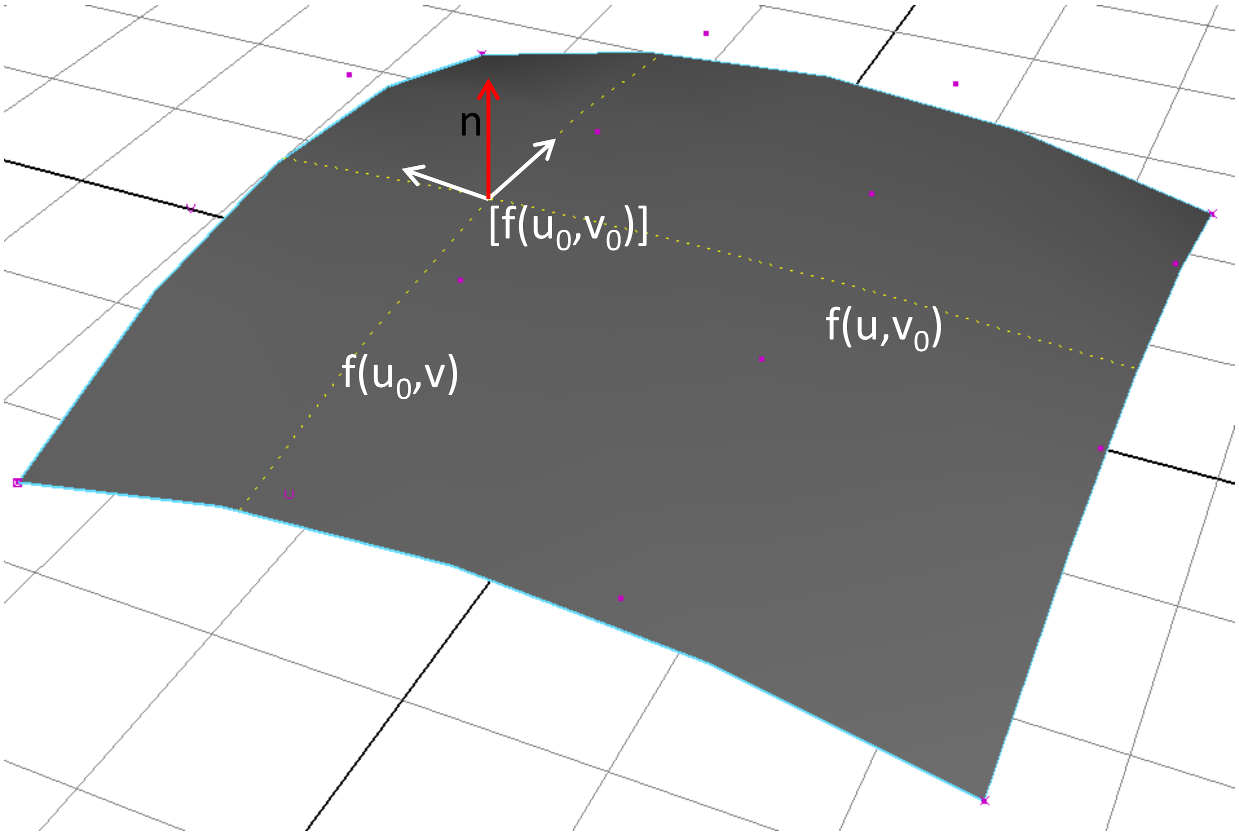


n=aXb

Normal vector of a parametric surface



Normal vector of a parametric surface



$$n = f'(u_0, v) \times f'(u, v_0)$$

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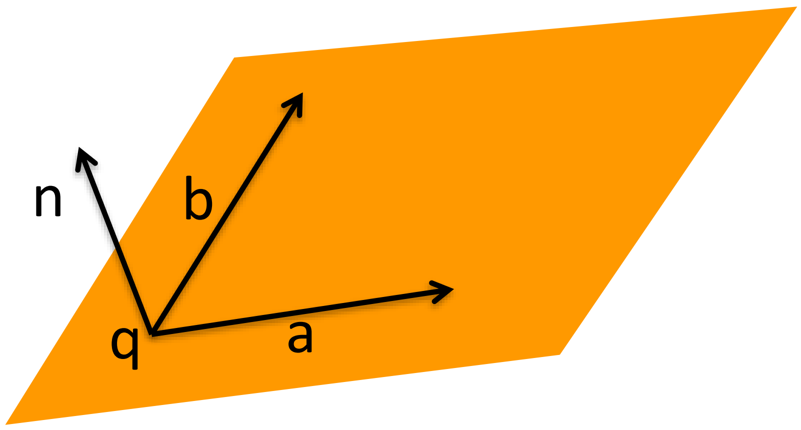
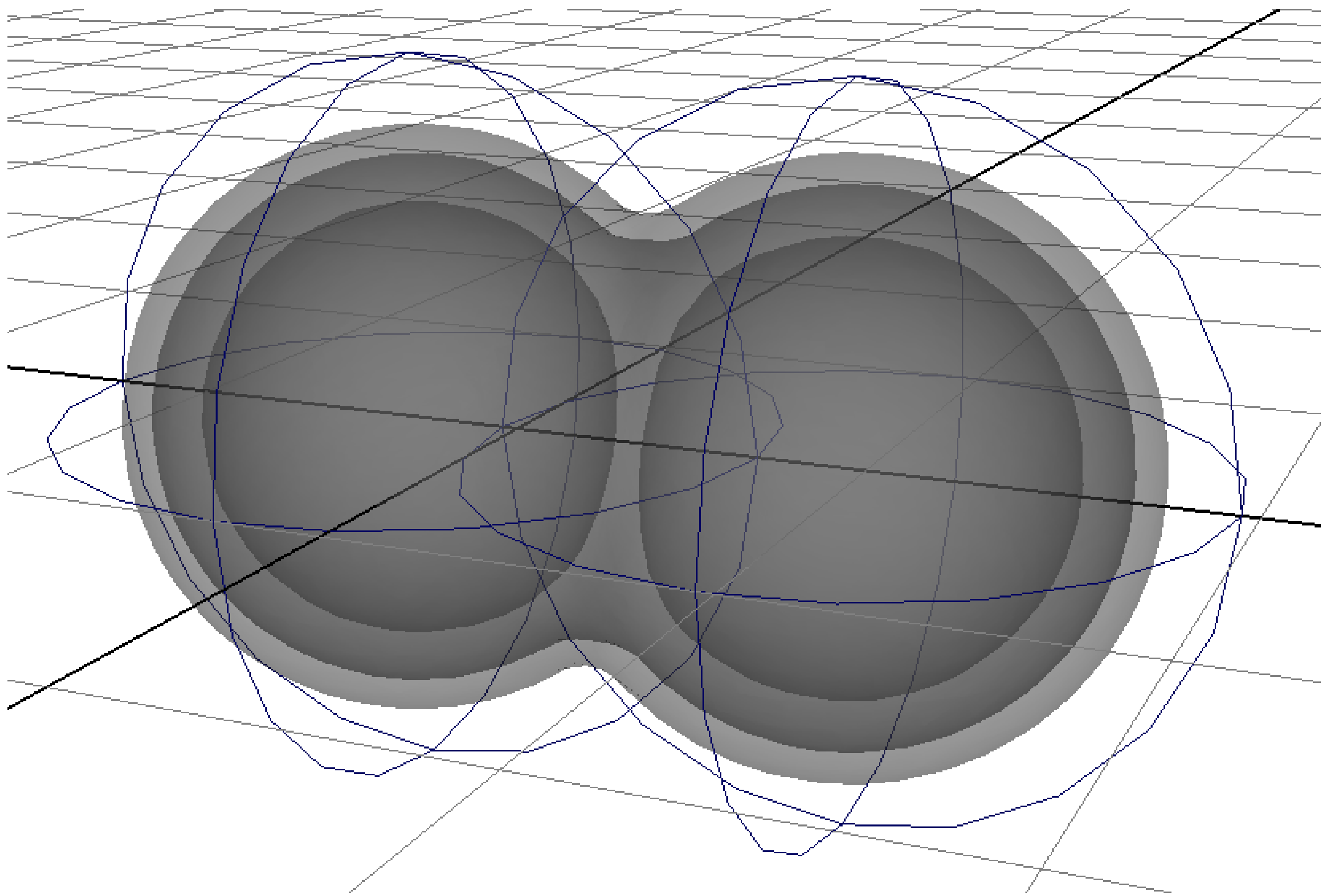
Implicit function of a plane



$$f(p) = (p-q).n=0$$

Implicit function: level sets



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3D parametric surfaces

- Extrude
- Revolve
- Loft
- Square

Demo…

3D parametric surfaces: Coons interpolation